A DIGITAL QUADRATURE AMPLITUDE MODULATION (QAM) RADIO

"Building a better radio ..."

- ⋆ Carrier Recovery
- ⋆ Baud Timing
- ⋆ Equalization
- ⋆ Prototype

QAM Radio (cont'd)

Coming Attractions:

- ⋆ Improved bandwidth utilization of quadrature modulation (QM)
- ⋆ Quadrature modulated PAM
- ⋆ Phase modulation as QM
- ⋆ Carrier offset impairments for QM
- ⋆ Costas loop for 4-QAM
- ⋆ Phase recovery ambiguity resolution
- ⋆ Quadruple frequency carrier extraction from fourth-power of QAM signal
- ⋆ Phase-locked-loop for 4-QAM
- ⋆ Constellation design for higher-order QAM
- ⋆ Power optimization timing for QAM
- ⋆ "Complex" equalization for QAM
- ⋆ Various QAM receiver architectures
- ⋆ QPSK prototype

QAM Radio (cont'd)

Reference Texts:

- J. B. Anderson, *Digital Transmission Engineering*, Prentice Hall, 1999.
- J. A. C. Bingham, *The Theory and Practice of Modem Design*, Wiley, 1988.
- R. D. Gitlin, J. F. Hayes, and S. B. Weinstein, *Data Communication Principles*, Plenum Press, 1992.
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- H. Meyr, M. Moeneclaey, and S. A. Fechtel, *Digital Communication Receivers: Synchronization, Channel Estimation, and Signal Processing*, Wiley, 1998.
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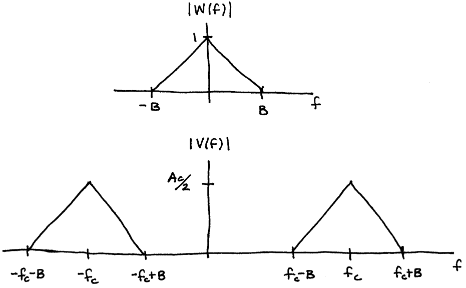
Improved Bandwidth Utilization of Quadrature Modulation (QM)

- One problem with double sideband AM (aka AM with suppressed carrier) of the real message signal $w(t)$ (with even symmetric magnitude spectrum) into the passband signal

$$v(t) = A_c w(t) \cos(2\pi f_c t)$$

is that $V(f)$ has twice the bandwidth of $W(f)$.



Improved Bandwidth Utilization of QM (cont'd)

- Quadrature modulation (QM) sends two message signals in same $2B$ passband bandwidth using orthogonal carriers cosine and sine

$$v(t) = A_c[m_1(t)\cos(2\pi f_c t + \theta) - m_2(t)\sin(2\pi f_c t + \theta)]$$

- The phase offset of a carrier does not effect the frequency translation of the baseband magnitude spectrum (or the resultant passband bandwidth), so upconverted m_1 and m_2 magnitude spectra will both be centered at (and even symmetric about) f_c.

Quadrature Modulated PAM

- The two message signals in QM could be composed as pulse-amplitude-modulated (PAM) signals

$$m_i(t) = \sum_k s_i[k]p(t - kT)$$

where $s_i[k]$ is the symbol sequence of the ith message drawn from a finite alphabet (e.g. ± 1, ± 3), T is the symbol interval, and $p(t)$ is the (time-limited) pulse-shape.

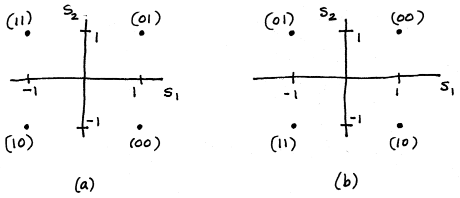
- The resulting transmitted quadrature modulated signal (with $A_c = 1$) is

$$v(t) = \sum_k p(t - kT)[s_1[k]\cos(2\pi f_c t + \theta) \\ -s_2[k]\sin(2\pi f_c t + \theta)]$$

where θ is the fixed (arbitrary) transmitter carrier phase.

Quadrature Modulated PAM (cont'd)

- The two data streams could be considered separate messages or a combined message. In either case, we will presume that the two data streams are both of zero average and are such that (e.g. uncorrelated) their average product is zero.
- The alphabet constellation can be plotted in a two-dimensional plane as a combined message. With each s_i binary (± 1), the four possible pairs $(s_1, s_2) =$ (1,-1), (1,1), (-1,1), (-1,-1) could be associated with the four pairs possible with 2 bits (00, 01, 11, 10) in different ways, e.g.



Phase Modulation as QM

- Consider a pulse-phase-modulated sequence

$$v(t) = g \sum_k p(t - kT)\cos(2\pi f_c t + \gamma(t))$$

where g is a fixed scaling gain and γ is a time-varying phase signal

$$\gamma(t) = \alpha[k] \quad kT \leq t < (k+1)T$$

with $\alpha[k]$ chosen from a set of, e.g., four possibilities: $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.

- Four phase choices could be associated with the four pairs of two bits (00, 10, 11, 01) in a conversion from message bits to transmitted signal.
- This phase modulation with 4 choices is called quadrature phase shift keying (QPSK).

Phase Modulation as QM (cont'd)

- Recall

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

so

$$\cos(2\pi f_c t + \gamma(t)) = \cos(2\pi f_c t)\cos(\gamma(t)) - \sin(2\pi f_c t)\sin(\gamma(t))$$

- With $g = \sqrt{2}$,

$$g\cos([\pi/4, 3\pi/4, 5\pi/4, 7\pi/4]) = [1, -1, -1, 1]$$
$$g\sin([\pi/4, 3\pi/4, 5\pi/4, 7\pi/4]) = [1, 1, -1, -1]$$

and

$$\cos(2\pi f_c t + \gamma(t)) = \pm\cos(2\pi f_c t) \pm \sin(2\pi f_c t))$$

- The resulting QPSK signal can be described as

$$\sum_k p(t-kT)[s_1[k]\cos(2\pi f_c t) - s_2[k]\sin(2\pi f_c t)]$$

with $s_i = \pm 1$, i.e. 4-QAM with $\theta = 0$.

Carrier Offset Impairments for QM

- To start, we will review ideal demodulation using sine and cosine mixer (with frequency and phase matching that at transmitter) each followed by LPF

⊙ $x_1(t) = v(t)\ \cos(2\pi f_c t)$

$$\begin{aligned} &= A_c m_1(t)\ \cos^2(2\pi f_c t) - A_c m_2(t)\ \sin(2\pi f_c t)\ \cos(2\pi f_c t) \\ &= \frac{A_c m_1(t)}{2}(1 + \cos(4\pi f_c t)) - \frac{A_c m_2(t)}{2}(\sin(4\pi f_c t)) \end{aligned}$$

⊕ $s_1(t) = \text{LPF}\{x_1(t)\} = \frac{A_c m_1(t)}{2}$

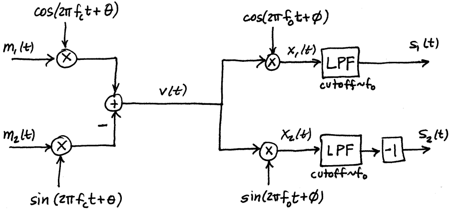
⊙ $x_2(t) = v(t)\ \sin(2\pi f_c t)$

$$\begin{aligned} &= A_c m_1(t)\ \cos(2\pi f_c t)\ \sin(2\pi f_c t) - A_c m_2(t)\ \sin^2(2\pi f_c t) \\ &= \frac{A_c m_1(t)}{2}\ \sin(4\pi f_c t) - \frac{A_c m_2(t)}{2}(1 - \cos(4\pi f_c t)) \end{aligned}$$

⊕ $s_2(t) = \text{LPF}\{-x_2(t)\} = \frac{A_c m_2(t)}{2}$

Carrier Offset Impairments for QM (cont'd)

- Presume receiver downconverter specification of frequency and phase offset from actual carrier frequency and phase.



- Transmitted signal

$$v(t) = m_1(t)\cos(2\pi f_c t + \theta) - m_2(t)\sin(2\pi f_c t + \theta)$$

- Downconverted signal on "cosine" path $x_1(t) = v(t)\cos(2\pi f_0 t + \phi)$

$$= m_1(t)\cos(2\pi f_c t + \theta)\cos(2\pi f_0 t + \phi)$$

$$-m_2(t)\sin(2\pi f_c t + \theta)\cos(2\pi f_0 t + \phi)$$

Carrier Offset Impairments for QM (cont'd)

- Downconverted signal on "sine" path $x_2(t) = v(t)\sin(2\pi f_0 t + \phi)$

$$= m_1(t)\cos(2\pi f_c t + \theta)\sin(2\pi f_0 t + \phi)$$
$$-m_2(t)\sin(2\pi f_c t + \theta)\sin(2\pi f_0 t + \phi)$$

- Recall

$$\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$
$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$
$$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

- So

$$x_1(t) = (1/2)m_1(t)\{\cos(2\pi(f_c - f_0)t + \theta - \phi)$$
$$+\cos(2\pi(f_c + f_0)t + \theta + \phi)\} - (1/2)m_2(t)\{\sin(2\pi(f_c - f_0)t + \theta - \phi)$$
$$+\sin(2\pi(f_c + f_0)t + \theta + \phi)\}$$

Carrier Offset Impairments for QM (cont'd)

- Similarly

$$x_2(t) = (1/2)m_1(t)\{\sin(2\pi(f_0 - f_c)t + \phi - \theta) + \sin(2\pi(f_c + f_0)t + \theta + \phi)\}$$
$$-(1/2)m_2(t)\{\cos(2\pi(f_c - f_0)t + \theta - \phi) - \cos(2\pi(f_c + f_0)t + \theta + \phi)\}$$

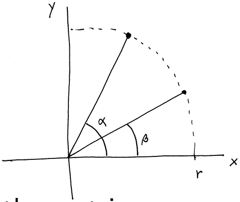
- Recall $\sin(-x) = -\sin(x)$
- With $f_c + f_0 \approx 2f_0$, for a LPF with cutoff well below $2f_0$

$$\text{LPF}\{x_1(t)\} = (1/2)m_1(t)\cos(2\pi(f_c - f_0)t + \theta - \phi)$$
$$-(1/2)m_2(t)\sin(2\pi(f_c - f_0)t + \theta - \phi)$$

$$\text{LPF}\{-x_2(t)\}$$
$$= (1/2)m_1(t)\sin(2\pi(f_c - f_0)t + \theta - \phi)$$
$$+(1/2)m_2(t)\cos(2\pi(f_c - f_0)t + \theta - \phi)$$

Carrier Offset Impairments for QM (cont'd)

- Consider two points in (x, y)-space on a circle of radius r. One is at angle α and the other at angle β with $\alpha > \beta$.



- We wish to confirm that the matrix

$$R = \begin{bmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{bmatrix}$$

rotates the point at radius r and angle β to the point at radius r and angle α, i.e. that

$$\begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} r = R \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} r$$

Carrier Offset Impairments for QM (cont'd)

- Assuming $r \neq 0$, reduces our objective to confirmation of two equations: $\cos(\alpha) = \cos(\alpha-\beta)\cos(\beta) - \sin(\alpha-\beta)\sin(\beta)$
$$\sin(\alpha) = \sin(\alpha-\beta)\cos(\beta) + \cos(\alpha-\beta)\sin(\beta)$$
- Recall:
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$
$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$
$$\cos^2(x) + \sin^2(x) = 1$$
- So: $\cos(\alpha-\beta)\cos(\beta) - \sin(\alpha-\beta)\sin(\beta)$
$$= (\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))\cos(\beta)$$
$$-(\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta))\sin(\beta)$$
$$= \cos(\alpha)\cos^2(\beta) + \sin(\alpha)\sin(\beta)\cos(\beta)$$
$$-\sin(\alpha)\sin(\beta)\cos(\beta) + \sin^2(\beta)\cos(\alpha)$$
$$= \cos(\alpha)(\sin^2(\beta) + \cos^2(\beta)) = \cos(\alpha)$$

Carrier Offset Impairments for QM (cont'd)

- Similarly
  $\sin(\alpha-\beta)\cos(\beta)+\cos(\alpha-\beta)\sin(\beta)$
  $$\begin{aligned} &= (\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta))\cos(\beta) \\ &+(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta))\sin(\beta) \\ &= \sin(\alpha)\cos^2(\beta)-\cos(\alpha)\cos(\beta)\sin(\beta) \\ &+\cos(\alpha)\cos(\beta)\sin(\beta)+\sin^2(\beta)\sin(\alpha) \\ &= \sin(\alpha)(\sin^2(\beta)+\cos^2(\beta)) = \sin(\alpha) \end{aligned}$$
- This confirms that pre-multiplication of an $[x \quad y]^T$-vector by the matrix
  $$R = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$$
  rotates the $[x \quad y]^T$ vector about the origin by the angle ψ.
- If ψ is positive the rotation is counterclockwise, or if $\psi < 0$ rotation is clockwise.

Carrier Offset Impairments for QM (cont'd)

- Recall

$$s_1(t) = \text{LPF}\{x_1(t)\}$$
$$= (1/2)m_1(t)\cos(\psi(t)) - (1/2)m_2(t)\sin(\psi(t))$$
$$s_2(t) = \text{LPF}\{-x_2(t)\}$$
$$= (1/2)m_1(t)\sin(\psi(t)) + (1/2)m_2(t)\cos(\psi(t))$$

where $\psi(t) = 2\pi(f_c - f_0)t + \theta - \phi$

- Gather into a single matrix equation

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cos(\psi(t)) & -\sin(\psi(t)) \\ \sin(\psi(t)) & \cos(\psi(t)) \end{bmatrix}\begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$$
$$= \frac{1}{2}R(t)\begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$$

Carrier Offset Impairments for QM (cont'd)

- The relationship

$$\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = \frac{1}{2} R(t) \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix}$$

with

$$R(t) = \begin{bmatrix} \cos(\psi(t)) & -\sin(\psi(t)) \\ \sin(\psi(t)) & \cos(\psi(t)) \end{bmatrix}$$

and

$$\psi(t) = 2\pi(f_c - f_0)t + \theta - \phi$$

reveals that the alteration due to carrier offset is a rotation of $(s_1(t), s_2(t))$ at a particular t by $2\pi(f_0 - f_c)t + \theta - \phi$.

- When $f_0 = f_c$ but $\phi \neq \theta$, the tilt of the $(s_1(t), s_2(t))$ relative to the message $(m_1(t), m_2(t))$ is fixed.
- When $f_0 \neq f_c$, $(s_1(t), s_2(t))$ is spinning relative to $(m_1(t), m_2(t))$.

Costas Loop for 4-QAM

- Sampling perfectly downconverted 4-QAM signals $s_1(t)$ and $s_2(t)$ at the proper times should produce one of four pairs $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$.
- For this time-synchronized sampled 4-QAM constellation, a rotation of an integer multiple of 90° in the carrier recovery offset $(2\pi(f_c - f_0)t + \theta - \phi)$ will produce samples at the alphabet values.
- To exploit this symmetry, we will extend the Costas loop for PAM to 4-QAM by seeking a scheme that causes the carrier recovery offset to converge to an integer multiple of 90° (where for PAM, the carrier recovery offset was designed to converge to an offset of an integer multiple of 180°). We will resolve this ambiguity later.

Costas Loop for 4-QAM (cont'd)

- Our objective is to adjust the receiver mixer phase $\phi(t)$ to assure

$$\phi(t) = 2\pi(f_c - f_0)t + \theta + \rho\pi/2$$

  where ρ is a fixed integer.
- We will begin by assuming that $f_c = f_0$ and θ is fixed but unknown, so our objective is $\phi = \theta + \rho(\pi/2)$.
- Consider the cost function $J_C = \cos^2(2(\theta - \phi))$ which, given $\cos^2(x) = (1/2)(1 + \cos(2x))$, is $(1/2)(1 + \cos(4(\theta - \phi))$.
- J_C has a maximum of one whenever $\cos(4(\theta - \phi)) = 1$ or

$$4(\theta - \phi) = 0,\ 2\pi, 4\pi, ... \quad \Rightarrow \quad \phi = \theta + \rho(\pi/2)$$

  for ρ an integer, as desired.

Costas Loop for 4-QAM (cont'd)

- Our adaptive update would be

$$\phi[k+1] = \phi[k] + \mu \frac{\partial J_C}{\partial \phi}|_{\phi=\phi[k]}$$

- So, our algorithm development reduces to a need to generate $\frac{\partial J_C}{\partial \phi}|_{\phi=\phi[k]}$.
- Using $\frac{d}{dx}(\cos(y)) = -(\sin(y))\frac{dy}{dx}$

$$\frac{\partial J_C}{\partial \phi} = 2\cos(2(\theta-\phi))\frac{\partial \cos(2(\theta-\phi))}{\partial(2(\theta-\phi))} \cdot \frac{\partial(2(\theta-\phi))}{\partial \phi}$$

$$= 4\cos(2(\theta-\phi))\sin(2(\theta-\phi))$$

- So, we need to generate a signal proportional to the product of the cosine and sine of twice $\theta - \phi$.

Costas Loop for 4-QAM (cont'd)

- The received 4-QAM signal is

$$v(t) = m_1(t)\cos(2\pi f_c t + \theta) - m_2(t)\sin(2\pi f_c t + \theta)$$

  where each $m_i(t)$ is a binary PAM signal.
- Define the four signals

$$x_1(t) = LPF\{v(t)\cos(2\pi f_c t + \phi)\}$$
$$x_2(t) = LPF\{v(t)\cos(2\pi f_c t + \phi + \pi/4)\}$$
$$x_3(t) = LPF\{v(t)\cos(2\pi f_c t + \phi + \pi/2)\}$$
$$x_4(t) = LPF\{v(t)\cos(2\pi f_c t + \phi + 3\pi/4)\}$$

- Recall

$$\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$

  and

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

Costas Loop for 4-QAM (cont'd)

- We can manipulate x_1 to reveal

$$\begin{aligned} x_1(t) &= LPF\{m_1(t)\cos(2\pi f_c t + \theta)\cos(2\pi f_c t + \phi) \\ &\quad - m_2(t)\sin(2\pi f_c t + \theta)\cos(2\pi f_c t + \phi)\} \\ &= LPF\{(1/2)(m_1(t)[\cos(\theta - \phi) + \cos(4\pi f_c t + \theta + \phi)] \\ &\quad - m_2(t)[\sin(\theta - \phi) + \sin(4\pi f_c t + \theta + \phi)]\} \\ &= (1/2)[m_1(t)\cos(\theta - \phi) - m_2(t)\sin(\theta - \phi)] \end{aligned}$$

- Similarly,

$$\begin{aligned} x_2(t) &= (1/2)[m_1(t)\cos(\theta - \phi - (\pi/4)) \\ &\quad - m_2(t)\sin(\theta - \phi - (\pi/4))] \\ x_3(t) &= (1/2)[m_1(t)\cos(\theta - \phi - (\pi/2)) \\ &\quad - m_2(t)\sin(\theta - \phi - (\pi/2))] \\ x_4(t) &= (1/2)[m_1(t)\cos(\theta - \phi - (3\pi/4)) \\ &\quad - m_2(t)\sin(\theta - \phi - (3\pi/4))] \end{aligned}$$

Costas Loop for 4-QAM (cont'd)

- Now form the product
$x_1(t)x_3(t) = (1/4)[m_1(t)\cos(\theta-\phi) - m_2(t)\sin(\theta-\phi)]$

$$\cdot[m_1(t)\cos(\theta-\phi-(\pi/2)) - m_2(t)\sin(\theta-\phi-(\pi/2))]$$

$$= (1/4)[m_1^2(t)\cos(\theta-\phi)\cos(\theta-\phi-(\pi/2))$$

$$+m_2^2(t)\sin(\theta-\phi)\sin(\theta-\phi-(\pi/2))$$

$$-m_1(t)m_2(t)\cos(\theta-\phi)\sin(\theta-\phi-(\pi/2))$$

$$-m_1(t)m_2(t)\sin(\theta-\phi)\cos(\theta-\phi-(\pi/2))]$$

$$= (1/8)[m_1^2(t)(\cos(\pi/2) + \cos(2(\theta-\phi)-(\pi/2))$$

$$+m_2^2(t)(\cos(\pi/2) - \cos(2(\theta-\phi)-(\pi/2))$$

$$-m_1(t)m_2(t)(\sin(-\pi/2) + \sin(2(\theta-\phi)-(\pi/2))$$

$$-m_1(t)m_2(t)(\sin(\pi/2) + \sin(2(\theta-\phi)-(\pi/2))]$$

- Because $\cos(\frac{\pi}{2}) = 0$, $\sin(\frac{\pm\pi}{2}) = \pm 1$, $\sin(x) = \cos(x-\frac{\pi}{2})$, and $\cos(x) = -\sin(x-\frac{\pi}{2})$

$$x_1(t)x_3(t) = (1/8)[(m_1^2(t) - m_2^2(t))\sin(2(\theta-\phi))$$

$$+2m_1(t)m_2(t)\cos(2(\theta-\phi))]$$

Costas Loop for 4-QAM (cont'd)

- Similarly manipulate $x_2(t)x_4(t)$

$$
\begin{aligned}
x_2(t)x_4(t) = (1/4)[m_1(t)\cos(\theta-\phi-(\pi/4)) \\
-m_2(t)\sin(\theta-\phi-(\pi/4))] \\
\cdot[m_1(t)\cos(\theta-\phi-(3\pi/4)) \\
-m_2(t)\sin(\theta-\phi-(3\pi/4))] \\
=(1/4)[m_1^2(t)\cos(\theta-\phi-(\pi/4))\cos(\theta-\phi-(3\pi/4)) \\
+m_2^2(t)\sin(\theta-\phi-(\pi/4))\sin(\theta-\phi-(3\pi/4)) \\
-m_1(t)m_2(t)\cos(\theta-\phi-(\pi/4))\sin(\theta-\phi-(3\pi/4)) \\
-m_1(t)m_2(t)\sin(\theta-\phi-(\pi/4))\cos(\theta-\phi-(3\pi/4))] \\
=(1/8)[m_1^2(t)(\cos(\pi/2)+\cos(2(\theta-\phi)-\pi) \\
+m_2^2(t)(\cos(\pi/2)-\cos(2(\theta-\phi)-\pi) \\
-m_1(t)m_2(t)(\sin(-\pi/2)+\sin(2(\theta-\phi)-\pi) \\
-m_1(t)m_2(t)(\sin(\pi/2)+\sin(2(\theta-\phi)-\pi)]
\end{aligned}
$$

Costas Loop for 4-QAM (cont'd)

- Because $\cos(\pi/2) = 0$, $\sin(\pm\pi/2) = \pm 1$, $\sin(x - \pi) = -\sin(x)$, and $\cos(x - \pi) = -\cos(x)$,

$$x_2(t)x_4(t) = (1/8)[-(m_1^2(t) - m_2^2(t))\cos(2(\theta - \phi)) \\ +2m_1(t)m_2(t)\sin(2(\theta - \phi))]$$

- And now form

$$x_1(t)x_2(t)x_3(t)x_4(t) = (1/64)[(m_1^2(t) - m_2^2(t))\sin(2(\theta - \phi)) \\ +2m_1(t)m_2(t)\cos(2(\theta - \phi))] \\ \cdot[-(m_1^2(t) - m_2^2(t))\cos(2(\theta - \phi)) \\ +2m_1(t)m_2(t)\sin(2(\theta - \phi))] \\ = (1/64)[(-(m_1^2(t) - m_2^2(t))^2 + 4m_1^2(t)m_2^2(t)) \\ \cdot\sin(2(\theta - \phi))\cos(2(\theta - \phi)) \\ +2m_1(t)m_2(t)(m_1^2(t) - m_2^2(t)) \\ \cdot(\sin^2(2(\theta - \phi)) - \cos^2(2(\theta - \phi)))]$$

Costas Loop for 4-QAM (cont'd)

- Because
$$-(m_1^2(t) - m_2^2(t))^2 - 4m_1^2(t)m_2^2(t)$$
$$= -m_1^4(t) - 2m_1^2(t)m_2^2(t) - m_2^4(t)$$
$$= -(m_1^2(t) + m_2^2(t))^2$$
and from
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
the four-term product becomes $x_1(t)x_2(t)x_3(t)x_4(t)$
$$= (8m_1^2(t)m_2^2(t) - (m_1^2(t) + m_2^2(t))^2)$$
$$\cdot \sin(2(\theta - \phi)) \cos(2(\theta - \phi))$$
$$-2m_1(t)m_2(t)(m_1^2(t) - m_2^2(t)) \cos(4(\theta - \phi))$$
- Recall that we are attempting to produce a signal proportional to $\sin(2(\theta - \phi)) \cos(2(\theta - \phi))$ to use as the gradient term in the adaptation of ϕ.

Costas Loop for 4-QAM (cont'd)

- For the moment, consider any pulse shape, such as a rectangle or the Hamming blip, that is time-limited to within one symbol interval of T seconds, i.e. $p(x) = 0$ for $x < 0$ or $x > T$.
- Recall
$$m_i(t) = \sum_k s_i[k] p(t - kT)$$
where $s_i[k] = 1$ or -1.
- For T-wide pulse shapes
$$m_i^2(t) = \sum_k s_i^2[k] p^2(t - kT)$$
- For 4-QAM
$$m_i^2(t) = \sum_k p^2(t - kT) = \eta(t)$$
which is periodic with period T.

Costas Loop for 4-QAM (cont'd)

- Thus,

$$m_1^2(t) - m_2^2(t) = 0$$
$$m_1^2(t)m_2^2(t) = \eta^2(t)$$
$$m_1^2(t) + m_2^2(t) = 2\eta(t)$$

- So, the four-term product $x_1(t)x_2(t)x_3(t)x_4(t)$

$$= (8m_1^2(t)m_2^2(t) - (m_1^2(t) + m_2^2(t))^2) \cdot \sin(2(\theta - \phi)) \cos(2(\theta - \phi))$$
$$-2m_1(t)m_2(t)(m_1^2(t) - m_2^2(t)) \cos(4(\theta - \phi))$$
$$= 4\eta^2(t) \sin(2(\theta - \phi)) \cos(2(\theta - \phi))$$

which is proportional via a nonnegative factor to $\sin(2(\theta - \phi)) \cos(2(\theta - \phi))$.

- In this special case using a symbol interval limited pulse shape, we can update ϕ via

$$\phi[k+1] = \phi[k] + \mu x_1(t)x_2(t)x_3(t)x_4(t)|_{t=kT_s,\phi=\phi[k]}$$

with T_s the sample period (typically less than the symbol period T).

Costas Loop for 4-QAM (cont'd)

- We can also resort to an explanation based on averaging the four-term product, as the adaptive algorithm will.
- As the basic approximate gradient descent update is a lowpass integration akin to a short-term time average, the proportionality of the created signal to the desired $\sin(2(\theta - \phi)) \cos(2(\theta - \phi))$ need only be true for the average of the four-term product.
- We assume that $m_1(t)$ and $m_2(t)$ are such that their individual averages are zero and the average of the product of two terms each composed of only one m_i is the product of their averages. For example, the average of $m_i m_j^3$ (for $i \neq j$) equals the average of m_i times the average of m_j^3. Because the average of m_i is zero, such a term would vanish on average.

Costas Loop for 4-QAM (cont'd)

- This reduces the average of the four-term product to $\text{avg}\{x_1(t)x_2(t)x_3(t)x_4(t)\}$
$$= \text{avg}\{(8m_1^2(t)m_2^2(t) - (m_1^2(t) + m_2^2(t))^2)\}$$
$$\cdot \sin(2(\theta - \phi))\cos(2(\theta - \phi))$$
- With the average over time of $m_i^2(t) = \alpha$ and the average of $m_i^4(t) = \beta$ (with β not necessarily equal to α^2) $\text{avg}\{(8m_1^2(t)m_2^2(t) - (m_1^2(t) + m_2^2(t))^2)\}$
$$= 6\alpha^2 - 2\beta$$
- So, without regard for the specific pulse shape, the average of $x_1(t)x_2(t)x_3(t)x_4(t)$ is proportional to $\sin(2(\theta - \phi))\cos(2(\theta - \phi))$ (as long as $3\alpha^2 \neq \beta$) and therefore suitable for use in the quadriphase Costas loop adaptation law (as presented above in conjunction with the symbol-time-limited pulse shape case).

Costas Loop for 4-QAM (cont'd)

- We will now test

$$\text{avg}\{(8m_1^2(t)m_2^2(t) - (m_1^2(t) + m_2^2(t))^2)\}$$

numerically using cos4qam

```
s1=sign(rand([1,100])-0.5);  %symbols in message 1
s2=sign(rand([1,100])-0.5);  %symbols in message 2
N=length(s1);

% zero pad T-spaced symbol sequence to create
% upsampled T/M-spaced sequence of scaled
% T-spaced pulses (with T = 1 time unit)
M=100; mup1=zeros(1,N*M); mup1(1:M:end)=s1;
mup2=zeros(1,N*M); mup2(1:M:end)=s2;

unp=ones(1,M);                   %unnormalized pulse
p=sqrt(M)*unp/sqrt(sum(unp.^2)); %normalized pulse shape
m1=filter(p,1,mup1);     %convolve pulse shape with data
m2=filter(p,1,mup1);     %convolve pulse shape with data

yo=8*sum((m1.^2).*(m2.^2))/length(m1);
yoyo=sum(((m1.^2)+(m2.^2)).^2)/length(m1);
average_scale_factor=yo-yoyo
terms_ratio=yo/yoyo
```

Costas Loop for 4-QAM (cont'd)

- We can try other pulse shapes by replacing the line defining the unnormalized rectangular pulse shape

```
unp=ones(1,M);
```

  with a Hamming blip

```
unp=hamming(M);
```

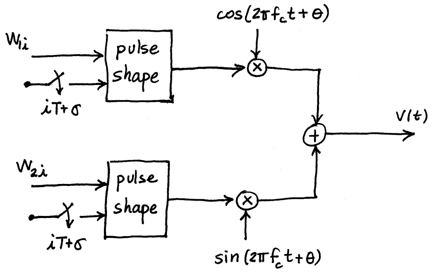
  or a square root raised cosine 20 symbols wide with a rollof factor of 0.3

```
unp=srrc(10,0.3,M,0);
```

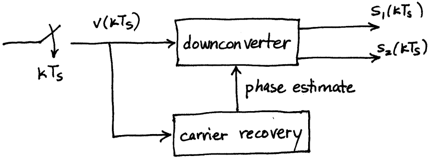
- In every case, the average scale factor is positive and the ratio of $8m_1^2(t)m_2^2(t)$ to $(m_1^2(t) + m_2^2(t))^2$ is always 2 to 1, as expected in our case where $(\text{avg}(m_i^2(t)))^2 = \text{avg}(m_i^4(t))$.

Costas Loop for 4-QAM (cont'd)

- QAM transmitter

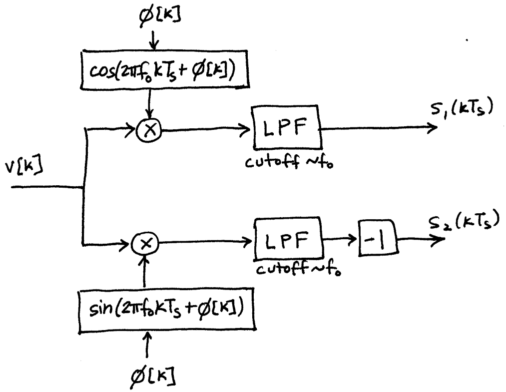


- QAM downconverter with carrier recovery



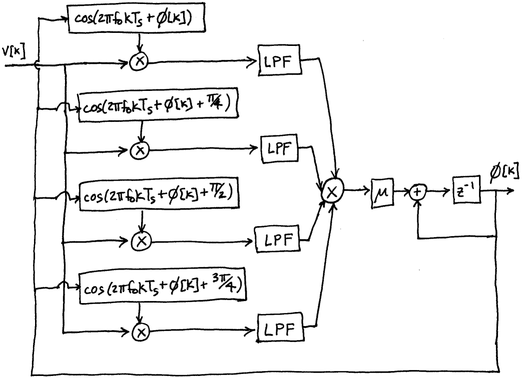
Costas Loop for 4-QAM (cont'd)

- QAM downconverter



Costas Loop for 4-QAM (cont'd)

- 4-QAM Costas loop carrier recovery



- Variants on this quadriphase Costas loop appear in chapter 6 of Bingham and section 4.2.3 of Anderson.

Phase Recovery Ambiguity Resolution

- We can resolve the phase ambiguity by
  - checking the demodulated sampled signal phase/polarity against a known/training signal or
  - differentially encoding the message source so the information is carried in how the successive symbols change (or not) from one sample to the next or
  - letting a trained equalizer automatically add a rotational phase to achieve a match to the training symbols
- Our QPSK Prototype Receiver includes the first and the last methods. Correlation of the in-phase training signal with both the in-phase and quadrature downsampled signals will reveal which is stronger and of what polarity for correction prior to passing signals on to the equalizer. The trained equalizer is not expected to be left with a phase ambiguity to resolve.

Phase Recovery Ambiguity Resolution (cont'd)

- Regarding differential coding, consider 2-PAM alphabet ± 1. Send $+1$ if current symbol same as previous; -1 if different.
- Sample sequence:
  1, -1, -1, 1, -1, -1, 1, 1, 1, ...
- Differentially encoded sequence (given knowledge of starting value of 1):
  ?, -1, 1, -1, -1, 1, -1, 1, 1, ...
- Decoding reverses process given knowledge of starting value of 1.
- An extension to 4-QAM is described in conjunction with Figure 16-4 in Lee and Messerschmitt.
- An isolated error in one symbol in a differentially encoded sequence will cause 2 symbol errors in recovered sequence.

Quadruple Frequency Carrier Extraction from Fourth-power of 4-QAM Signal

- As an alternative to the Costas loop, we return to the approach of attempting to extract a replica of the carrier from the received signal and track that.
- For double sideband PAM we preprocessed the received signal by squaring it and narrowly bandpass filtering about twice the carrier frequency to extract a signal proportional to the cosine of twice the carrier frequency with twice the carrier phase.
- For 4-QAM we will take the fourth power of the received/transmitted signal and use a narrow bandpass filter about 4 times the carrier frequency to extract a cosine with 4 times the carrier freqeuncy and phase.

Quadruple Frequency Carrier Extraction from Fourth-power of 4-QAM Signal (cont'd)

- Presume the transmitted signal

$$v(t) = m_1(t)\cos(2\pi f_c t + \theta) - m_2(t)\sin(2\pi f_c t + \theta)$$

is received without any additional interference.

- For notational convenience, we will write the received signal (which matches the transmitted v) as

$$v = a\cos(\gamma) - b\sin(\gamma)$$

with a for $m_1(t)$, b for $m_2(t)$, and γ for $2\pi f_c t + \theta$.

- Squaring v

$$v^2 = a^2\cos^2(\gamma) - 2ab\cos(\gamma)\sin(\gamma) + b^2\sin^2(\gamma)$$

Quadruple Frequency Carrier Extraction from Fourth-power of 4-QAM Signal (cont'd)

- Using $\cos^2(\gamma)$, $\sin^2(\gamma)$, and $\sin(\gamma)\cos(\rho)$ formulas

$$v^2 = a^2(1/2)(1+\cos(2\gamma))$$
$$-2ab(1/2)(\sin(0)+\sin(2\gamma))$$
$$+b^2(1/2)(1-\cos(2\gamma))$$
$$= (1/2)(a^2-b^2)\cos(2\gamma) + (1/2)(a^2+b^2)$$
$$-ab\sin(2\gamma)$$

- Why not use v^2?
- We could bandpass filter at $2f_c$ (with $\gamma = 2\pi f_c t + \theta$), which rejects the DC term $(1/2)(a^2+b^2)$.
- With a and b representing uncorrelated, zero-mean signals average$(ab) = 0$. The zero DC content of ab implies that a sufficiently narrow BPF will remove $ab\sin(2\gamma)$ as well.

Quadruple Frequency Carrier Extraction from Fourth-power of 4-QAM Signal (cont'd)

- Similarly, with a and b and identically distributed, the average of a^2 matches that of b^2 and the remaining term will also have zero average and be removed by a narrow bandpass filter.
- Nothing is left after narrowly bandpass filtering v^2, so we are on to squaring v^2 and again using $\cos^2$, $\sin^2$, and $\sin \cdot \cos$ formulas

$$
\begin{aligned}
v^4 &= (1/4)(a^2-b^2)^2\cos^2(2\gamma) + (1/4)(a^2+b^2)^2 \\
&+a^2b^2\sin^2(2\gamma) + (1/2)(a^2-b^2)(a^2+b^2)\cos(2\gamma) \\
&-ab(a^2-b^2)\cos(2\gamma)\sin(2\gamma) - ab(a^2+b^2)\sin(2\gamma) \\
&= (1/8)(a^2-b^2)^2(1+\cos(4\gamma)) + (1/4)(a^2+b^2)^2 \\
&+(1/2)a^2b^2(1-\cos(4\gamma)) \\
&+(1/2)(a^2-b^2)(a^2+b^2)\cos(2\gamma) \\
&-(1/2)ab(a^2-b^2)\sin(4\gamma) - ab(a^2+b^2)\sin(2\gamma)
\end{aligned}
$$

Quadruple Frequency Carrier Extraction from Fourth-power of 4-QAM Signal (cont'd)

- Bandpass filtering narrowly about 4γ will remove all terms not multiplying either $\sin(4\gamma)$ or $\cos(4\gamma)$.
- In our previous discussion of v^2 we noted that a and b represent zero-mean, uncorrelated signals expected to have an average product of zero. So, a sufficiently narrow bandpass filtering will remove the term including $\sin(4\gamma)$.
- Thus, narrow bandpass filtering of v^4 will leave

$$\begin{aligned}
&\{(1/8)(a^2 - b^2)^2 - (1/2)a^2b^2\} \cos(4\gamma) \\
&= \{(1/8)(a^4 - 2a^2b^2 + b^4) - (1/2)a^2b^2\} \cos(4\gamma) \\
&= \{(1/8)(a^4 + 2a^2b^2 + b^4) - (1/2)a^2b^2 \\
&\quad -(1/2)a^2b^2\} \cos(4\gamma) \\
&= -\{(1/8)(8a^2b^2 - (a^2 + b^2)^2)\} \cos(4\gamma)
\end{aligned}$$

Quadruple Frequency Carrier Extraction from Fourth-power of 4-QAM Signal (cont'd)

- For the independent, identically distributed m_1 and m_2 represented by a and b, we previously (in the 4-QAM Costas loop derivation) utilized (and verified numerically) the fact that the term inside the braces is positive for 4-QAM.
- Thus, we have extracted a signal proportional (through a nonpositive scale factor) to the quadrupled carrier with four times the frequency and four times the phase.
- If f_0 at the receiver is used to specify center frequency of BPF, we should redefine γ as $2\pi f_0 t + \theta + 2\pi(f_c - f_0)t = 2\pi f_0 t + \theta(t)$. If the bandwidth of the BPF is wide enough, $\cos(4\gamma)$ will still be passed. But, if too wide, unwanted extraneous signals will be passed too. Thus, we have a design tradeoff between uncertain f_c and a desired narrow BPF.

Phase-Locked Loop for QAM

- Having extracted a sampled sinusoid scaled by a time-varying gain (that is on average sign definite) at a multiple (4) of the carrier frequency and phase, we are ready to apply this signal to a phase tracking loop such as a digital phase-locked loop (PLL).
- To have built a digital BPF to extract samples of a cosine at $4f_0$, sampling must occur above the Nyquist rate for $4f_0$ (i.e. $8f_0$) or undesirable aliasing will occur.
- The fourth-power PLL will use as cost function

$$(1/4)\cos(4(\theta - \phi))$$

  which is to be maximized by choice of ϕ as θ.

Phase-Locked Loop for QAM (cont'd)

- An approximate gradient descent strategy updates ϕ via

$$\phi[k+1] = \phi[k] + \mu \frac{\partial (1/4)\cos(4(\theta - \phi))}{\partial \phi}|_{\phi=\phi[k]}$$

$$= \phi[k] + \mu(1/4)\{\frac{\partial \cos(4(\theta-\phi))}{\partial(\theta-\phi)} \cdot \frac{\partial(\theta-\phi)}{\partial\phi}\}|_{\phi=\phi[k]}$$

$$= \phi[k] + \mu\{(-\sin(4(\theta-\phi)))(-1)|_{\phi=\phi[k]}$$

$$= \phi[k] + \mu \sin(4(\theta - \phi[k]))$$

- Our previous analysis suggests that the output (on average) of the BPF driven by the fourth-power of the received signal can be written as $-2g\cos(8\pi f_0 kT_s + 4\theta[k])$ where g is a nonnegative gain.

Phase-Locked Loop for QAM (cont'd)

- Given this BPF output, we can form the product

$$-2g\cos(8\pi f_0 kT_s + 4\theta)\sin(8\pi f_0 kT_s + 4\phi[k])$$

and using $\sin\cdot\cos$ formula reduce this to

$$-(1/2)2g(\sin(4(\phi[k]-\theta)) \\ +\sin(16\pi f_0 kT_s + 4\theta + 4\phi[k]))$$

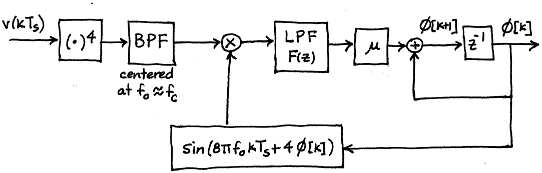
- Lowpass filtering this cosine-sine product with a transfer function the frequency response of which is flat with unit gain over its passband and using $\sin(-x) = -\sin(x)$ produces

$$LPF\{-2g\cos(8\pi f_0 kT_s + 4\theta) \\ \cdot\sin(8\pi f_0 kT_s + 4\phi[k])\} \\ \approx g\sin(4\theta - 4\phi[k])$$

This can be plugged in the adaptive update of ϕ for the approximate gradient.

Phase-Locked Loop for QAM (cont'd)

- The associated update of ϕ is diagrammed in



- Consider a non-ideal lowpass filter $F(z)$ that still passes no signals above some frequency below $4f_0$, but filters low frequency content through an impulse response $f[k]$ (with a potentially non-constant low frequency response)

$$gf[k] * \sin(4(\theta[k] - \phi[k]))$$

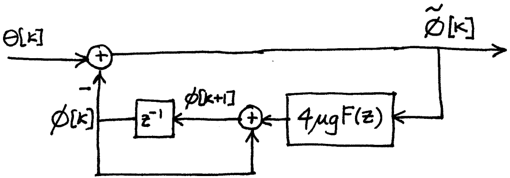
where $*$ indicates convolution.

Phase-Locked Loop for QAM (cont'd)

- With $\tilde{\phi} \equiv \theta - \phi$ and $\phi[k] \approx \theta$, $\sin(4\tilde{\phi}[k]) \approx 4\tilde{\phi}[k]$ so

$$gf[k] * \sin(4\tilde{\phi}[k]) \approx 4gf[k] * \tilde{\phi}[k]$$

- The resulting block diagram description with slowly time-varying $\theta[k]$ as the input and $\tilde{\phi}[k]$ as the output is



Phase-Locked Loop for QAM (cont'd)

- To reduce this block diagram to a transfer function as a single ratio of two polynomials in z, we will use the basic feedback loop reduction rule

$$y = f(r \pm gy) = fr \pm fgy$$

$$\Rightarrow \quad (1 \mp fg)y = fr \quad \Rightarrow \quad y = \left(\frac{f}{1 \mp fg}\right) r$$

- Thus, the transfer function from the output of the block with transfer function $4\mu g F(z)$ to $\phi[k]$ is $z^{-1}/(1 - z^{-1})$ or $1/(z-1)$.
- The transfer function from $\tilde{\phi}$ to ϕ is $4\mu g F(z)/(z-1)$.
- With $F(z)$ a ratio of two polynomials $F_N(z)/F_D(z)$, the transfer function from θ to $\tilde{\phi}$ is

$$\frac{1}{1 + \left(\frac{4\mu g F_N(z)}{(z-1)F_D(z)}\right)} = \frac{(z-1)F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

Phase-Locked Loop for QAM (cont'd)

- To evaluate the asymptotic output of this transfer function to a specific input, we will use the final value theorem.
- Final value theorem (of z-transforms):

$$\lim_{k\to\infty} x[k] = \lim_{z\to 1}(1 - z^{-1})X(z)$$

  where $\mathcal{Z}\{x[k]\} = X(z)$ and $(1 - z^{-1})X(z)$ has all poles strictly inside the unit circle in the z-plane.
- For $\theta[k]$ a step with height α, i.e. 0 for $k < 0$ and α for $k \geq 0$, $\mathcal{Z}\{\theta[k]\} = \alpha z/(z-1)$ so

$$\mathcal{Z}\{\tilde{\phi}[k]\} = \frac{\alpha z}{z-1}\frac{(z-1)F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

$$= \frac{\alpha z F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

Phase-Locked Loop for QAM (cont'd)

- Applying the final value theorem

$$\lim_{k\to\infty} \tilde{\phi}[k] = \lim_{z\to 1} \frac{z-1}{z} \frac{\alpha z F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

$$= \lim_{z\to 1} \frac{\alpha (z-1) F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

As long as $F_N(1) \neq 0$, $\lim_{k\to\infty} \tilde{\phi}[k] = 0$.

- For $\theta[k]$ a ramp with slope α, i.e. 0 for $k < 0$ and αk for $k \geq 0$, $\mathcal{Z}\{\theta[k]\} = \alpha z/(z-1)^2$ so

$$\mathcal{Z}\{\tilde{\phi}[k]\} = \frac{\alpha z}{(z-1)^2} \frac{(z-1)F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

$$= \frac{\alpha z F_D(z)}{(z-1)((z-1)F_D(z) + 4\mu g F_N(z))}$$

Phase-Locked Loop for QAM (cont'd)

- Applying the final value theorem
  $\lim_{k\to\infty} \tilde{\phi}[k]$

$$= \lim_{z\to 1} \frac{z-1}{z} \frac{\alpha z F_D(z)}{(z-1)((z-1)F_D(z) + 4\mu g F_N(z))}$$

$$= \lim_{z\to 1} \frac{\alpha F_D(z)}{(z-1)F_D(z) + 4\mu g F_N(z)}$$

$$= \frac{\alpha F_D(1)}{4\mu g F_N(1)}$$

  as long as $F_N(1) \neq 0$ and the roots of
  $\Delta(z) = (z-1)F_D(z) + 4\mu g F_N(z)$ are strictly inside the unit circle.
- If $F_D(1) \neq 0$, then $\lim_{k\to\infty} \tilde{\phi}[k]$ is nonzero and gets smaller with larger μ.
- If $F_D(z) = (z-1)$ so $F_D(1) = 0$, then $\lim_{k\to\infty} \tilde{\phi}[k]$ is zero.

Phase-Locked Loop for QAM (cont'd)

- The choice of $F(z)$ which includes a pole at $z = 1$ (aka an integrator) results in a "type II" PLL. The name arises for its ability to track a type 1 polynomial ($i = 1$ in input αk^i with k the time index) with zero error asymptotically and a type 2 polynomial (αk^2) with asymptotically constant (and finite) offset.
- Changing μ will shift closed-loop poles. Closer to the origin of the z-plane means faster decay of the transient response. Outside the unit circle means instability.
- Poles closer to the z-plane origin than $z = 1$ de-emphasize lowpass nature of θ to $\tilde{\phi}$ transfer function.

Phase-Locked Loop for QAM (cont'd)

- With $F(z) = b/(z-a)$,

$$\Delta(z) = (z-1)F_D(z) + 4\mu g F_N(z)$$
$$= (z-1)(z-a) + 4\mu g b$$

  As μ increases, the roots become complex with constant real part and increasing imaginary part that takes them outside the unit circle.

- With $F(z) = b/(z-1)$,

$$\Delta(z) = (z-1)^2 + 4\mu g b$$

  For all positive μ the closed loop poles are a complex conjugate pair with unity real part and complex portion proportional to μ, i.e. always unstable for $\mu > 0$.

Phase-Locked Loop for QAM (cont'd)

- With $F(z) = b(z-c)/(z-1)$,

$$\Delta(z) = (z-1)^2 + 4\mu g b(z-c)$$

  - In particular with $4\mu g b = 2$ and $c = 0.5$, $\Delta(z) = z^2$ which puts the closed-loop poles at the z-plane origin.
  - Alternatively, with $4\mu g b = 0.3$ and $c = 0.6$, $\Delta(z) = z^2 - 1.7z + 0.82$ which puts the closed-loop poles at $z \approx 0.85 \pm j0.31$.
  - For any c, sufficiently large $\mu g b$ will cause a root of $\Delta(z)$ to be outside the unit circle.

Phase-Locked Loop for QAM (cont'd)

- We have limited our investigation of fourth-power PLL carrier recovery primarily to 4-QAM. However, our ultimate design objective will be 16-QAM. As an indication of the application of this fourth-power PLL approach to 16-QAM, we cull some comments from some of our reference texts.
- From p. 149 of Anderson:
  "When the ... modulation is, for instance 16-QAM instead of QPSK, the PLL reference signal is not a steady $\cos(4\omega_0 t + 4\psi_0)$ and the phase difference signal has other components besides $\sin(4\psi_0 - 4\theta_0)$. Nonetheless, the synchronizer still works passably well."

Phase-Locked Loop for QAM (cont'd)

- From p. 173 of Bingham (1988):
  "This simple method [PLL tracking of quadrupled frequency] can be used for constellations with 16 points ... but it has been generally assumed that the pattern jitter would be intolerable. However, it can be shown that, at least for 16QAM, the outer points dominate, and the fourth-power signal has a component at $4f_c$ that is usable if a very narrow band PLL is used to refine it. Whether the independence from data decisions and equalizer convergence that this forward-acting method offers outweighs the problems of such a narrow-band PLL remains to be seen."

QAM Constellation Design

- Assuming that soft decision symbol errors in recovering each message pair $(s_1[k], s_2[k])$ are circularly distributed about each constellation point makes the minimum distance between any two points an important indicator of hard decision symbol error susceptibility.
- If the symbol errors are (circularly) uniformly distributed over a fixed radius, until the maximum of this range exceeds half the minimum distance between any two constellation points, a nearest element decision device will sustain no hard decision symbol errors.
- If the symbol errors are circularly gaussian, hard decision errors will always occur due to the presumed unbounded nature of the gaussian "noise", but are infrequent if the variance of this noise is much less than half the minimum distance between any 2 points in the constellation.

QAM Constellation Design (cont'd)

- For 4-QAM, the constellation should be a square to avoid one pair of symbol errors (on the otherwise shorter side of the rectangle) becoming more likely.
- From p. 422 of Proakis and Salehi, the probability of an incorrect decision or symbol error rate (SER) for an M-QAM system is

$$1 - \left[1 - \left(1 - \frac{1}{\sqrt{M}} \right) \operatorname{erfc} \left(\sqrt{\frac{3\sigma_s^2}{2(M-1)\sigma_a^2}} \right) \right]^2$$

where σ_s^2 is the variance of the white, zero-mean, symbol sequence and σ_a^2 is the variance of the sum of everything (including ISI and noise gain) that causes the soft decision not to be a constellation point (and is assumed to be circularly gaussian distributed).

QAM Constellation Design (cont'd)

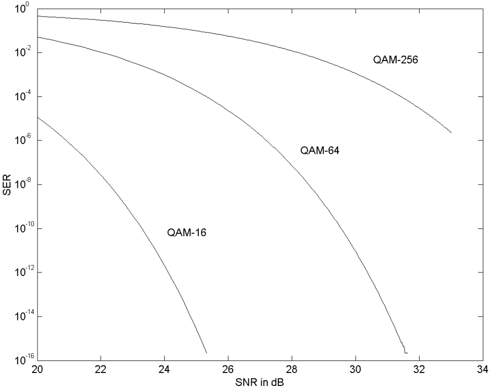
- "erfc" function in this SER formula is the complementary error function (see help *erfc* in Matlab).
- Because erfc(x) monotonically decreases as x increases,
  - As M increases with SNR ($= \sigma_s^2/\sigma_a^2$) unaltered, symbol error rate (SER), i.e.

$$1 - \left[1 - \left(1 - \frac{1}{\sqrt{M}}\right) \mathrm{erfc}\left(\sqrt{\frac{3\sigma_s^2}{2(M-1)\sigma_a^2}}\right)\right]^2$$

    increases.
  - As SNR increases (because σ_s^2 increases or σ_a^2 decreases) with M unaltered, SER decreases.

QAM Constellation Design (cont'd)

- The following plot of SER versus SNR ($= \sigma_s^2/\sigma_a^2$) for square constellations 4-QAM, 16-QAM, and 256-QAM confirms the need for higher SNR to achieve the same bit error rate with a higher-order QAM constellation.



QAM Constellation Design (cont'd)

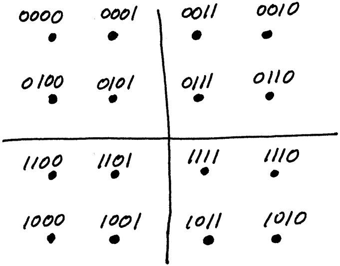
- Preceding plot from `sersnr`

```
snr=[100:10:2000];
M=[16 64 256];
for Mind=1:length(M)
for snrind=1:length(snr)
yo=1-(1/sqrt(M(Mind)));
efa=sqrt(3*snr(snrind)/(2*(M(Mind)-1)));
ser(Mind,snrind)=1-(1-yo*erfc(efa))^2; end end
semilogy(10*log10(snr),ser','k')
```

- A crude SNR estimate (actually an upper bound) is available from eye diagram and its cluster variance, i.e. average squared difference between soft and associated hard decisions.
- To inhibit common symbol errors from turning into multiple bit errors, we could try to keep the bit changes between symbols at the minimum distance from each other in the constellation to only one if possible. For 4-QAM, consider $45^\circ \to 00$, $135^\circ \to 01$, $-135^\circ \to 11$, and $-45^\circ \to 10$.

QAM Constellation Design (cont'd)

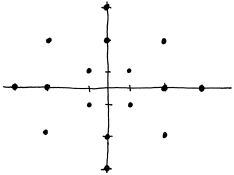
- A mapping from the constellation points to the symbols that minimizes adjacent symbol errors is termed a Gray coding of the data bits. For example, for 16QAM



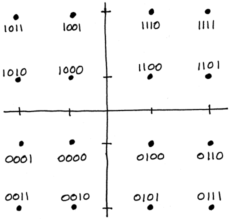
- Could choose to omit corner points to reduce maximum to minimum signal power range over which system analog electronics must retain linearity.
- Non-square QAM constellations can prove simpler for synchronization but exhibit higher SER for same SNR than square constellations.

QAM Constellation Design (cont'd)

- From p. 97 of Anderson: The V.29 modem standard "has worse error performance in AWGN than does ... [square] 16-QAM, but it is easier to synchronize"
- V.29



- V.32alt



Power Optimization Timing for QAM

- We presume successful digital downconversion has occured with carrier recovery, producing two sampled sequences each with the same fixed sample period and same timing offset in the designation of sample index zero relative to a universal clock.
- Both baseband signals are presumed to have the same bandwidth. The fixed sampler period is shorter than the maximum possible satisfying the Nyquist condition on the bandlimited baseband pulse-shaped signal.
- This oversampled sequence will be filtered by the matched filter associated with the transmitter pulse shape.

Power Optimization Timing (cont'd)

- The matched filter output will be put into an interpolator designed to extract a baud-spaced sequence of samples with the synchronization of these symbol-sample times to the zero index of the universal clock off by some time increment (that is some fraction of a symbol period).
- For a PAM signal (as we saw in *Software Receiver Design*), the selection of this baud-timing increment can be done by optimizing the average of the absolute value or the 2nd or 4th powers of the resulting baud-spaced sequence values.
- Whether we minimize or maximize depends on the pulse shape and the power used.

Power Optimization Timing (cont'd)

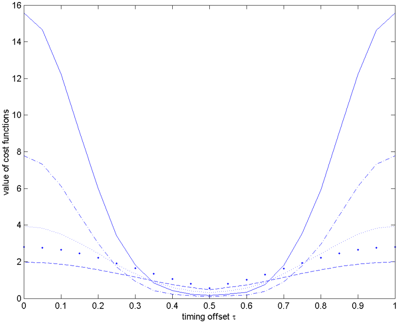
- With successfully downconverted QAM, we have two (presumably independent) PAM signals with the same timing offset.
- With real PAM, optimizing the average of the absolute value of the single PAM signal raised to the 1st, 2nd, or 4th power can produce desirable timing selection.
- For QAM, we could consider the sum of the average of the 1st, 2nd, or 4th powers of the absolute values of the two constituent signals s_1 and s_2, i.e. the average of $|s_1|^n + |s_2|^n$ for $n = 1$, 2, 4.
- Or we could consider the average of the 1st, 2nd, or 4th powers of the length of the vector in the 2-dimensional I-Q space, i.e. the average of $(|\sqrt{(s_1^2 + s_2^2)}|)^n$ for $n = 1$, 2, 4.
- For $n = 2$ and our presumption that s_1 and s_2 are white and uncorrelated with each other, these two performance measures are the same.

Power Optimization Timing (cont'd)

- To check their utility, draw these candidate costs with `basetimcost` with a 20 times oversampled hamming pulse shape.

```
unp=hamming(m);
cp=conv(unp,unp);       % pulse shape combo
ps=sqrt(m)*cp/sqrt(sum(cp.^2));
cost1=zeros(1,m+1); cost2=zeros(1,m+1);
cost3=zeros(1,m+1); cost4=zeros(1,m+1);
cost5=zeros(1,m+1);
n=1000; x=zeros(1,n);   % "monte carlo" method
for i=1:m+1             % for each offset
% create +/-1 sequence
s1=sign(rand([1,n/m])-0.5); s2=sign(rand([1,n/m])-0.5);
% zero pad T-spaced symbol sequence to create
% upsampled T/m-spaced sequence of scaled T-spaced pulses
mup1=zeros(1,n); mup1(1:m:end)=s1;
mup2=zeros(1,n); mup2(1:m:end)=s2;
m1=filter(ps,1,mup1);   % convolve pulse shape with data
m2=filter(ps,1,mup2);   % convolve pulse shape with data
%sampled baseband data with timing offset iT/m
sm1=m1((length(ps)-1)/2+i:m:end);
sm2=m2((length(ps)-1)/2+i:m:end); end
cost1(i)=sum(sqrt(sm1.^2+sm2.^2))/length(sm1);  % abs
cost2(i)=sum(sm1.^2+sm2.^2)/length(sm1);      % square
cost3(i)=sum((sm1.^2+sm2.^2).^2)/length(sm1); % 4th pow
cost4(i)=sum(abs(sm1)+abs(sm2))/length(sm1); % sum of abs
cost5(i)=sum(sm1.^4+sm2.^4)/length(sm1);     % sum of 4th
end
```

Power Optimization Timing (cont'd)



- dashed: absolute value (average $|\sqrt{(s_1^2 + s_2^2)}|$)
- dotted (little dots): square (average $(|\sqrt{(s_1^2 + s_2^2)}|)^2$)
- solid: 4th power (average $(|\sqrt{(s_1^2 + s_2^2)}|)^4$)
- dotted (big dots): sum of absolute values (average $|s_1| + |s_2|$
- dash-dot: sum of 4th powers (average $|s_1|^4 + |s_2|^4$)

Power Optimization Timing (cont'd)

- For these plots, the zero sample was set to be half the length of the pulse shape, which is the desired setting.
- The fourth power curve has the steepest decline from the desired answer and therefore should exhibit fastest convergence.
- The same curves can be drawn for a square root raised cosine pulse shape. The combination of the transmit and matched receive filter will produce a raised cosine "transfer function", and just as observed in the PAM case in *Software Receiver Design*, the 1st and 2nd power curves will require maximization, while minimization is to be used with the 4th power.

Power Optimization Timing (cont'd)

- Assuming the desire for minimization of the the sum of the average fourth powers of the two baseband signal streams (already successfully downconverted from the received RF signal), we can form the approximate gradient descent algorithm
$\tau[k+1] = \tau[k]$

$$-\bar{\mu}\frac{\partial(1/4)(m_1^4(kT+\tau)) + m_2^4(kT+\tau)}{\partial\tau}|_{\tau=\tau[k]}$$

$$= \tau[k] - \bar{\mu}[(m_1^3(kT+\tau) + m_2^3(kT+\tau))$$

$$\cdot\frac{\partial(m_1(kT+\tau) + m_2(kT+\tau))}{\partial\tau}]|_{\tau=\tau[k]}$$

- If we were pursuing maximization, the minus in front of the positive $\bar{\mu}$ would be a plus.

Power Optimization Timing (cont'd)

- As in *Software Receiver Design*, we can use a numerical estimate of the derivative of $m_i(kT+\tau)$ with respect to τ, resulting in

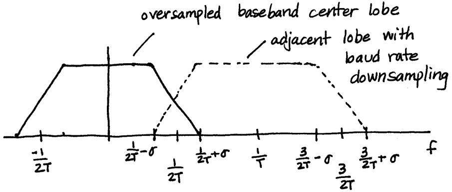
$$\tau[k+1] = \tau[k] - \mu(m_1^3(kT+\tau[k]) + m_2^3(kT+\tau[k]))$$
$$\cdot[m_1(kT+\tau+\delta) - m_1(kT+\tau-\delta) + m_2(kT+\tau+\delta)$$
$$-m_2(kT+\tau-\delta)]$$

  where $\mu = \bar{\mu}/\delta$ and δ is small and positive.

- All of the values for the $m_i(t)$ for the offsets $\tau[k]$, $\tau[k]-\delta$, and $\tau[k]+\delta$ are computed from the available oversampled m_i via interpolation.

Power Optimization Timing (cont'd)

- While the sampled baseband signal passing through the receiver matched filter and the timing recovery interpolators is presumed sampled well above its Nyquist frequency, this is not typically true for the downsampled to the symbol rate output of the timing recovery block preceding a baud-spaced equalizer.
- Consider a common square root raised cosine pulse shape with excess bandwidth (typically in the 10 to 50% range). The spectrum after downsampling will suffer aliasing in a limited region about the frequency $1/2T$ where T is the symbol period.



Power Optimization Timing (cont'd)

- If the timing recovery downsampler is to be followed by a baud-spaced equalizer, our biggest concern is with deep nulls in the apparent channel transfer function seen by the equalizer. Deep nulls are leveled out by large gains in a baud-spaced equalizer over the frequency band of the channel null. Any channel noise in this channel null band will be heavily amplified and degrade the capabilities of a memoryless decision device at the equalizer output.
- When the transmission channel is distortionless, the only frequencies over which such nulling might occur would be where the magnitudes of the overlapping segments of the total downsampled spectrum are comparable. This occurs only within a band of twice the excess bandwidth centered at $1/(2T)$.

Power Optimization Timing (cont'd)

- Can destructive cancellation occur with timing offset which does not effect the magnitude of the spectrum, only its phase, but in a frequency dependent way? Yes; adding aliased portions from the leading edge and the rear edge of the baseband center lobe that suffer sufficient relative phase shift can create a null. See section 7.5.1 of Bingham for such an example.
- Thus, a useful variant of power maximization for timing recovery pre-filters the downconverted signals with a bandpass filter centered at $1/2T$ with a bandwidth of twice (or less) the excess bandwidth of the pulse shape (from $(1/2T) - \sigma$ to $(1/2T) + \sigma$).
- As excess bandwidth approaches 100% of $1/T$, the BPF becomes an all-pass and does not effect the power optimization.

Power Optimization Timing (cont'd)

- This bandedge, bandpass filtering can effect the need for maximization or minimization, just as changing the pulse shape can.
- In terms of our adaptive algorithm, this BPF preprocesing can be seen as beneficial in its reduction of signals that would require reduced stepsizes (and the resulting slowed convergence) to accommodate in our approximate gradient descent scheme with modest timing jitter.
- From Lee and Messerschmitt, p. 745: "For some signals, particularly when the excess bandwidth is low, a fourth power nonlinearity ... is better than the magnitude squared. In fact, fourth-power timing recovery can even extract timing tones from signals with zero excess bandwidth. ... Simulations for QPSK ... suggest that fourth-power circuits out-perform absolute-value circuits for signals with less than about 20% excess bandwidth."

Power Optimization Timing (cont'd)

- From Lee and Messerschmitt, p. 745: "If timing recovery is done in discrete-time, aliasing must be considered... Any nonlinearity will increase the bandwidth of the ... signal ... In the presence of sampling, however, the high frequency components due to the nonlinearity can alias back into the bandwidth of the bandpass filter, resulting in additional timing jitter. ... Therefore, in a discrete-time realization, a magnitude-squared nonlinearity usually has a considerable advantage over either absolute-value or fourth-power nonlinearity."

Power Optimization Timing (cont'd)

- From Lee and Messerschmitt, p. 747: "The same relative merits of squaring, absolute-value, and fourth-power techniques apply to passband timing recovery as to baseband. In particular, absolute-value and fourth-power are usually better than squaring, except when aliasing is a problem in discrete-time implementations. As with baseband signals, it is sometimes advantageous to prefilter the signal before squaring."

"Complex" QAM Equalizer

- Consider this advice from Bingham, p. 231: "A complex equalizer ... can compensate for any demodulating carrier phase, but it is easier to deal with frequency offset by using a separate circuit or algorithm that, because it deals with only one variable, carrier phase, can move faster without causing jitter."
- To interpret this design guidance, we reconsider the rotation caused by carrier recovery offset for a distortionless channel. Here this situation is presumed to have been achieved by imperfect downconversion accompanied by acceptable baud-timing and equalization.

"Complex" QAM Equalizer (cont'd)

- For a recovered symbol pair vector $x = [x_1 \;\; x_2]^T$ off by a rotation of ψ radians from the "true" symbol pair vector $s = [s_1 \;\; s_2]^T$

$$x = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix} s = Rs$$

- Thus, we would like to recover s from x via a derotation by $-\psi$ or $\begin{bmatrix} \cos(-\psi) & -\sin(-\psi) \\ \sin(-\psi) & \cos(-\psi) \end{bmatrix} x$

$$= \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix} x = Px$$

- Confirm that

$$P = \begin{bmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{bmatrix} = R^{-1}$$

as expected, by showing that $PR = I$.

"Complex" QAM Equalizer (cont'd)

- The matrix multiplication Px can be written out as

$$\hat{s}_1 = \cos(-\psi)x_1 - \sin(-\psi)x_2$$

$$\hat{s}_2 = \sin(-\psi)x_1 + \cos(-\psi)x_2$$

  where the $\hat{s}_i$ are the recovered estimates of s_1 and s_2.
- Rather than interpret the pairs as vectors, we can consider them as the real and imaginary parts of a complex number, e.g. $s = s_1 + js_2$ and $x = x_1 + jx_2$.
- Consider multiplying the complex x by a complex gain f

$$(x_1 + jx_2)(f_1 + jf_2) = x_1 f_1 - x_2 f_2 + j(x_1 f_2 + x_2 f_1)$$

$$= \hat{s}_1 + j\hat{s}_2$$

  or

$$\hat{s}_1 = f_1 x_1 - f_2 x_2$$

$$\hat{s}_2 = f_2 x_1 + f_1 x_2$$

"Complex" QAM Equalizer (cont'd)

- This matches the format of the necessary rotation when $f_1 = \cos(-\psi)$ and $f_2 = \sin(-\psi)$.
- So, multiplying all of the (complex) equalizer coefficients by the same complex gain factor can correct for rotational offset, as noted by Bingham (before the "but" in the quotation above).
- If this rotational offset is changing with time, as we would expect with even a slight carrier frequency offset, then adjustment of this gain factor will require that all of the otherwise well-set equalizer gains move. This observation stimulates the second part of the quote from Bingham, which supports consideration of a single-complex-coefficient (trained or decision-directed) derotator after a complex equalizer.

"Complex QAM Equalizer (cont'd)

- Consider the two received signals, in-phase r_1 and quadrature r_2, composed as a complex number $r_1 + jr_2$ and multiplied by a (de)rotator $e^{j\theta}$ to produce $x_1 + jx_2$.
- Using $e^{jx} = \cos(x) + j\sin(x)$, form

$$x_1 + jx_2 = (r_1 + jr_2)e^{j\theta} = (r_1 + jr_2)(\cos(\theta) + j\sin(\theta))$$
$$= (r_1\cos(\theta) + j^2 r_2\sin(\theta) + j(r_2\cos(\theta) + r_1\sin(\theta))$$

So,

$$x_1 = r_1\cos(\theta) - r_2\sin(\theta)$$
$$x_2 = r_2\cos(\theta) + r_1\sin(\theta)$$

- The derotated signals x_1 and x_2 would be quantized to form hard decisions s_1 $(= \text{sign}(x_1))$ and s_2 $(= \text{sign}(x_2))$ or $s = \text{sign}(x)$ with $s = s_1 + js_2$.
- Consider as cost function the average of $(1/2)(s-x)(s-x)^*$ where the superscript $*$ indicates complex conjugation.

“Complex QAM Equalizer (cont'd)

- A stochastic gradient descent algorithm for adapting derotator angle θ is

$$\theta[k+1] = \theta[k] - \mu \frac{\partial (1/2)(s-x)(s-x)^*}{\partial \theta}|_{\theta=\theta[k]}$$

- Observe

$$\begin{aligned}
(s-x)(s-x)^* &= (s_1 + js_2 - x_1 - jx_2) \\
&\cdot(s_1 - js_2 - x_1 + jx_2) \\
&= s_1^2 - js_1s_2 - s_1x_1 + js_1x_2 + js_2s_1 - j^2s_2^2 \\
&-js_2x_1 + j^2s_2x_2 - x_1s_1 + js_2x_1 + x_1^2 - jx_1x_2 \\
&-jx_2s_1 + j^2x_2s_2 + jx_2x_1 - j^2x_2^2 \\
&= (s_1^2 - 2s_1x_1 + x_1^2) + (s_2^2 - 2s_2x_2 + x_2^2) \\
&= (s_1 - x_1)^2 + (s_2 - x_2)^2
\end{aligned}$$

- Thus,

$$\frac{\partial (1/2)(s-x)(s-x)^*}{\partial \theta} = (s_1 - x_1)(-1)\frac{\partial x_1}{\partial \theta} + (s_2 - x_2)(-1)\frac{\partial x_2}{\partial \theta}$$

“Complex QAM Equalizer (cont’d)

- Given $\frac{d}{dx}(\cos(x)) = -\sin(x)$ and $\frac{d}{dx}(\sin(x)) = \cos(x)$,

$$\frac{\partial x_1}{\partial \theta} = -r_1 \sin(\theta) - r_2 \cos(\theta) = -x_2$$

$$\frac{\partial x_2}{\partial \theta} = -r_2 \sin(\theta) + r_1 \cos(\theta) = x_1$$

- Thus,

$$\frac{\partial (1/2)(s-x)(s-x)^*}{\partial \theta} = (s_1 - x_1)x_2 - (s_2 - x_2)x_1$$

$$= s_1x_2 - x_1x_2 - s_2x_1 + x_2x_1 = s_1x_2 - s_2x_1$$

- The decision-directed adaptive derotator update algorithm for 4-QAM can be written as $\theta[k+1] = \theta[k] - \mu(s_1[k]x_2[k] - s_2[k]x_1[k])$ where $x[k] = r[k]e^{j\theta[k]}$ or

$$x_1[k] = r_1[k]\cos(\theta[k]) - r_2[k]\sin(\theta[k])$$

$$x_2[k] = r_2[k]\cos(\theta[k]) + r_1[k]\sin(\theta[k])$$

“Complex QAM Equalizer (cont’d)

- An alternative cost function can be based on dispersion minimization by considering as cost function the average of

$$((\mathrm{Re}(re^{-j\theta})^2 - \gamma)^2$$

where γ is a real constant and $\mathrm{Re}(a + jb) = a$.

- This dispersion cost function can be justified by visualizing the projection onto the real axis of a rotated, but otherwise perfect, symbol constellation relative to the single point of the positive real axis projection of a desirably squared-off perfect 4-QAM constellation.
- The associated dispersion-minimizing stochastic gradient descent scheme is

$$\theta[k+1] = \theta[k] - \mu((\mathrm{Re}(re^{-j\theta})^2 - \gamma)$$
$$\cdot \mathrm{Re}(re^{-j\theta[k]})\mathrm{Im}(re^{-j\theta[k]})$$

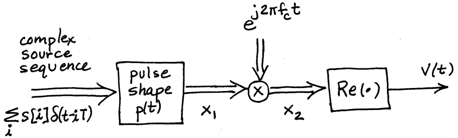
where $\mathrm{Im}(a + jb) = b$

"Complex" QAM Equalizer (cont'd)

- Having motivated the consideration of a complex signal model as mathematically convenient for the QAM derotation task, we continue with this complex system modelling with a "complex" model of real passband QAM signal creation

$$v(t) = \cos(2\pi f_c t) \sum_i \mathrm{Re}(s[i]) p(t - iT) - \sin(2\pi f_c t) \sum_i \mathrm{Im}(s[i]) p(t - iT)$$

- In the complex transmitter model



$$x_1(t) = \sum_i \mathrm{Re}(s[i]) p(t - iT) + j \sum_i \mathrm{Im}(s[i]) p(t - iT)$$

"Complex" QAM Equalizer (cont'd)

- Given $e^{jx} = \cos(x) + j\sin(x)$

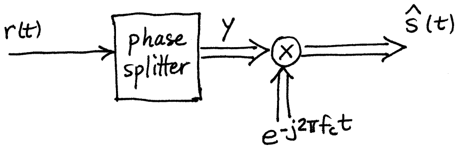
$$x_2(t) = \cos(2\pi f_c t)\sum_i \mathrm{Re}(s[i])p(t-iT)$$
$$+j\sin(2\pi f_c t)\sum_i \mathrm{Re}(s[i])p(t-iT)$$
$$+j\cos(2\pi f_c t)\sum_i \mathrm{Im}(s[i])p(t-iT)$$
$$+j^2\sin(2\pi f_c t)\sum_i \mathrm{Im}(s[i])p(t-iT)$$

- Thus, $\mathrm{Re}(x_2(t)) = v(t)$ and the complex source transmitter creates a real passband QAM signal.
- One model for a complex QAM downconverter model uses a phase splitter with impulse response $\phi(t)$ and Fourier transform

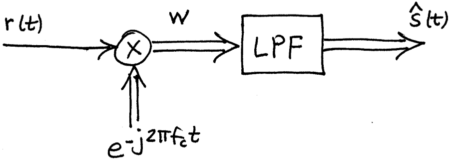
$$\Phi(f) = 1 \quad \text{for} \quad f \geq 0 \quad \text{and} \quad 0 \quad \text{for} \quad f < 0$$

"Complex" QAM Equalizer (cont'd)

- The phase splitter is followed by exponential multiplication

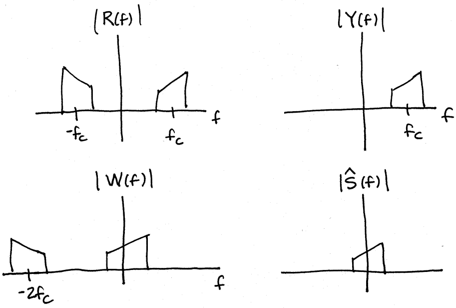


- An alternative complex QAM downconverter model follows exponential multiplication with a lowpass filter



"Complex" QAM Equalizer (cont'd)

- Their effects on the passband spectrum



Recall that multiplication by e^{jxt} shifts the entire spectrum to the left for $x < 0$.

- The baseband spectrum of the recovered complex source sequence is asymmetric, as expected. Symmetry about zero frequency would indicate a real baseband signal.

"Complex" QAM Equalizer (cont'd)

- To restore the asymmetric-baseband of the complex source model from its distortion by the commonly assymetric dynamics of a linear channel in the signal passband requires a complex equalizer.
- One exception is a passband channel with its real frequency portion symmetric about the carrier frequency which will result in a symmetric equivalent baseband model following successful downconversion.
- For a complex equalizer, computation of a nonnegative cost function utilizes, e.g., the product of the error and its complex conjugate rather than the straight square of its error. This results in adaptive updates that effectively add complex conjugations in the right places to adaptation laws for "real" equalizers.

"Complex" QAM Equalizer (cont'd)

- As an example, consider the complex version of LMS with training for adapting a linear combiner with training error

$$e[k] = d[k] - X^T[k]f$$

where

$$e[k] = e_R[k] + je_I[k]$$
$$d[k] = d_R[k] + jd_I[k]$$
$$X[k] = X_R[k] + jX_I[k]$$
$$f[k] = f_R[k] + jf_I[k]$$

- The stochastic gradient descent update minimizing the nonnegative $\text{avg}\{e[k]e^*[k]\}$ is

$$f[k+1] = f[k] - \bar{\mu}\{\frac{\partial}{\partial f_R}[e[k]e^*[k]]|_{f_R=f_R[k]} + j\frac{\partial}{\partial f_I}[e[k]e^*[k]]|_{f_I=f_I[k]}\}$$

where $*$ indicates complex conjugation.

"Complex" QAM Equalizer (cont'd)

- To facilitate algorithm derivation expand Xf as

$$X^T[k]f = (X_R^T[k] + jX_I^T[k])(f_R + jf_I)$$

$$= X_R^T[k]f_R + jX_I^T[k]f_R + jX_R^T[k]f_I$$

$$+j^2X_I^T[k]f_I$$

$$= (X_R^T[k]f_R - X_I^T[k]f_I) + j(X_I^T[k]f_R + X_R^T[k]f_I)$$

  which also yields $(X^T[k]f)^* = (X_R^T[k]f_R - X_I^T[k]f_I)$

$$-j(X_I^T[k]f_R + X_R^T[k]f_I)$$

- Recall that

$$e[k]e^*[k] = (e_R[k] + je_I[k])(e_R[k] - je_I[k])$$

$$= e_R^2[k] + je_I[k]e_R[k] - je_I[k]e_R[k] - j^2e_I^2[k]$$

$$= e_R^2[k] + e_I^2[k]$$

"Complex" QAM Equalizer (cont'd)

- So

$$\begin{aligned} e_R[k] = \text{Re}\{e[k]\} &= \text{Re}\{d[k]\} - \text{Re}\{X^T[k]f\} \\ &= d_R[k] - X_R^T[k]f_R + X_I^T[k]f_I \\ e_I[k] = \text{Im}\{e[k]\} &= \text{Im}\{d[k]\} - \text{Im}\{X^T[k]f\} \\ &= d_I[k] - X_I^T[k]f_R + X_R^T[k]f_I \end{aligned}$$

- With

$$\frac{\partial d_R}{\partial f_R} = \frac{\partial d_R}{\partial f_I} = \frac{\partial d_I}{\partial f_R} = \frac{\partial d_I}{\partial f_I} = 0$$

we can form

$$\frac{\partial}{\partial f_R}\{e[k]e^*[k]\} = \frac{\partial}{\partial f_R}\{e_R^2[k] + e_I^2[k]\}$$

$$= \frac{\partial}{\partial e_R[k]}\{e_R^2[k]\}\frac{\partial}{\partial f_R}\{e_R[k]\} + \frac{\partial}{\partial e_I[k]}\{e_I^2[k]\}\frac{\partial}{\partial f_R}\{e_I[k]\}$$

$$= -2e_R[k]X_R[k] - 2e_I[k]X_I[k]$$

"Complex" QAM Equalizer (cont'd)

- Similarly, with respect to the imaginary component of the equalizer parameter vector

$$\begin{aligned}\frac{\partial}{\partial f_I}\{e[k]e^*[k]\} &= \frac{\partial}{\partial f_I}\{e_R^2[k]+e_I^2[k]\}\\ &= \frac{\partial}{\partial e_R[k]}\{e_R^2[k]\}\frac{\partial}{\partial f_I}\{e_R[k]\}\\ &\quad+\frac{\partial}{\partial e_I[k]}\{e_I^2[k]\}\frac{\partial}{\partial f_I}\{e_I[k]\}\\ &= 2e_R[k]X_I[k]-2e_I[k]X_R[k]\end{aligned}$$

- Therefore,

$$\begin{aligned}f[k+1] = f[k]-\bar{\mu}\{&\frac{\partial}{\partial f_R}[e[k]e^*[k]]|_{f_R=f_R[k]}+j\frac{\partial}{\partial f_I}[e[k]e^*[k]]|_{f_I=f_I[k]}\}\\ = f[k]+\bar{\mu}(2)&(e_R[k]X_R[k]+e_I[k]X_I[k]\\ &+j(-e_R[k]X_I[k]+e_I[k]X_R[k]))\end{aligned}$$

"Complex" QAM Equalizer (cont'd)

- Because

$$e[k]X^*[k] = (e_R[k] + je_I[k])(X_R[k] - jX_I[k])$$

$$= e_R[k]X_R[k] + e_I[k]X_I[k]$$

$$+j(e_I[k]X_R[k] - e_R[k]X_I[k])$$

  we can write

$$f[k+1] = f[k] + \mu e[k]X^*[k]$$

- Relative to the real version of trained LMS, a complex conjugation has been added to the regressor vector. That is all.

"Complex" QAM Equalizer (cont'd)

- Fractionally-spaced equalizers can replace the sequential combination of an oversampled matched filter, an interpolator/downsampler, and baud-spaced equalizer.
- With downconversion split between a principal (but inexact) downconversion prior to equalization and a post-equalizer derotator the training/decision error used to adapt the equalizer must be rotated into the pre-derotated signal frame of the equalizer. This assumes that the rotation is slow enough that it is effectively constant over the time window covered by the equalizer's impulse response.

Various QAM Receiver Architectures

- Analog receiver (from chapter 5 of Bingham)
  1. bandpass filter
  2. downconversion with carrier recovery
  3. forward sampler adjustment
  4. decision device
  5. decoder
- Analog receiver with decision-directed carrier recovery (from chapter 5 of Bingham)
  1. bandpass filter
  2. automatic gain control
  3. downconversion with decision-directed carrier recovery
  4. forward sampler adjustment
  5. decision device
  6. decoder

Various QAM Receiver Architectures (cont'd)

- Nearly-all-digital receiver with linear equalizer (from chapter 5 of Bingham)

(1) free-running sampler

(2) bandpass filter

(3) phase splitter

(4) interpolator with feedback passband timing recovery

(5) automatic gain control with feedback from decision device

(6) preliminary free-running downconversion

(7) linear equalizer with post-derotator decision-directed adaptation

(8) carrier recovery via decision-directed rotation

Various QAM Receiver Architectures (cont'd)

- All-digital receiver with decision feedback equalizer (from chapter 5 of Bingham)

(1) free-running RF-sub-Nyquist sampler

(2) bandpass filter

(3) automatic gain control with feedback from decision device

(4) phase splitter

(5) interpolator with feedback passband timing recovery

(6) preliminary free-running downconversion

(7) linear equalizer with post-derotator decision-directed adaptation

(8) carrier recovery via decision-directed rotation

(9) decision feedback equalization adapted via decision-direction

Various QAM Receiver Architectures (cont'd)

- Typical Digital QAM Receiver (from section 4.1 of Meyr, Moeneclaey, and Fechtel)
  (1) analog preliminary downconversion with possible analog forward correction and/or digital feedback correction
  (2) free-running sampler
  (3) digital downconversion with frequency adjust
  (4) interpolator-matched filter-decimator
  (5) derotator
  (6) detection/decoding (also equalization)

Various QAM Receiver Architectures (cont'd)

- Generic equalized data demodulator (from Treichler, Larimore, and Harp, "Practical Demodulators for High-Order QAM Signals," *Proc. IEEE*, October 1998)
  (1) analog bandpass filter
  (2) analog automatic gain control
  (3) analog downconversion with decision-directed carrier tracking
  (4) sampler with timing recovery
  (5) equalizer with (hard or decoded) decision-directed adaptation
  (6) optional digital downconversion with decision-directed carrier recovery
  (7) optional decision feedback equalization
  (8) optional error-correcting decoding

Various QAM Receiver Architectures (cont'd)

- Generic demodulator for QAM signals (from Treichler, Larimore, and Harp, "Practical Demodulators for High-Order QAM Signals," *Proc. IEEE*, October 1998)
  (1) analog bandpass filter
  (2) analog AGC with feedback correction from free-running sampler output
  (3) quadrature free-running downconversion
  (4) resampler with "four-corners" timing recovery
  (5) FIR equalizer with selected blind adaptation algorithms
  (6) digital downconversion with decision-directed carrier recovery
  (7) decision device

Various QAM Receiver Architectures (cont'd)

- Multipoint network modem (from Jablon, "Joint Blind Equalization, Carrier Recovery, and Timing Recovery for High-Order QAM Signal Constellations," *IEEE Trans. on Signal Processing*, June 1992)
  1. sampler and AGC with adjusted sampler clock
  2. passband equalizer with de-spun error
  3. post-equalization carrier recovery
  4. decoding

Various QAM Receiver Architectures (cont'd)

- Typical passband QAM receiver (from section 6.4.6 of Lee and Messerschmitt)
  (1) analog bandpass filter
  (2) sampler with decision-directed adjustment
  (3) phase splitter
  (4) preliminary free-running downconversion
  (5) downsampling to $T/2$ sampling with decision-directed adjustment
  (6) fractionally-spaced linear (precursor) equalizer with de-spun decision-directed adjustment
  (7) digital downconversion with decision-directed carrier recovery
  (8) decision device
  (9) decision feedback (postcursor) equalizer with decision-directed adjustment

Various QAM Receiver Architectures (cont'd)

Some receiver architectures are more suited to particular operating circumstances than others.

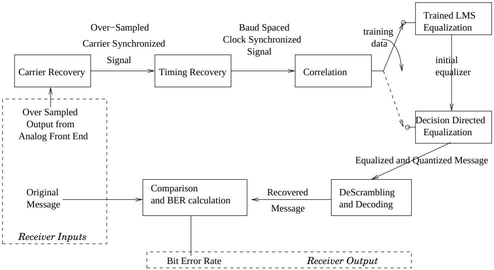
- From p. 731-2 of Lee and Messerschmitt:
  "One practical difficulty arises when an adaptive equalizer is used in conjunction with a decision-directed carrier recovery loop. Baseband adaptive equalizers assume that the input has been demodulated. The solution to this difficulty ... is to use a passband equalizer. ... By placing the forward equalizer before the carrier recovery demodulation, we avoid having the equalizer inside the carrier recovery loop. By contrast, a baseband equalizer would follow the demodulator and precede the slicer. This means that it is inside the carrier recovery loop. Consequently, the loop transfer function of the carrier recovery includes the time-varying equalizer, causing considerable complication."

Various QAM Receiver Architectures (cont'd)

- (continuing from p. 731-2 of Lee and Messerschmitt):
  "At the very least, the long delay (several symbol intervals) associated with the baseband equalizer would force the loop gain of the carrier recovery to be reduced to ensure stability, impairing its ability to track rapidly varying carrier phase. The passband equalizer ... mitigates this problem by equalizing prior to demodulation."
- From p. 429 of Gitlin, Hayes, and Weinstein:
  "At low SNR, the designer can of course use the nonlinear/PLL carrier recovery scheme ... but at moderate-to-high SNR levels, when data decisions reliably replicate the transmitted data, the data-directed loop has become the preferred carrier recovery system."

QPSK Prototype

- Possible received signal impairments
  - carrier freq and initial phase offset from specified values used in receiver
  - clock freq and initial phase offset from specified values used in receiver
  - phase noise random walk
  - timing offset random walk
  - multipath FIR channel
  - time varying multipath gains
  - wideband additive (gaussian) channel noise
- Receiver schematic



QPSK Prototype (cont'd)

- Receiver processing sequence
  (1) free-running 4 times oversampled (relative to baud interval) received passband signal
  (2) mixer with phase adaptation via dual quadriphase Costas loop
  (3) lowpass filtering for downconversion, matched filter, and interpolation all provided by matched filter with adjusted timing offset adapted with maximization of fourth power of downsampled signals in dual loop configuration
  (4) correlation used to resolve phase ambiguity and to locate training sequence start in equalizer input
  (5) linear equalizer adaptation via LMS; switched to decision-directed LMS adaptation during data (i.e. non-training) portion
  (6) frame-synchronized descrambler and (5,2) linear block code decoder