2

Uniform Plane Waves

2.1 Uniform Plane Waves in Lossless Media

The simplest electromagnetic waves are uniform plane waves propagating along some fixed direction, say the z-direction, in a lossless medium $\{\epsilon, \mu\}$.

The assumption of uniformity means that the fields have no dependence on the transverse coordinates x, y and are functions only of z, t. Thus, we look for solutions of Maxwell's equations of the form: $\boldsymbol{E}(x,y,z,t)=\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(x,y,z,t)=\boldsymbol{H}(z,t)$.

Because there is no dependence on x, y, we set the partial derivatives† $\partial_x = 0$ and $\partial_y = 0$. Then, the gradient, divergence, and curl operations take the simplified forms:

$$\nabla = \hat{\mathbf{z}}\frac{\partial}{\partial z}, \quad \nabla\cdot\boldsymbol{E} = \frac{\partial E_z}{\partial z}, \quad \nabla\times\boldsymbol{E} = \hat{\mathbf{z}}\times\frac{\partial \boldsymbol{E}}{\partial z} = -\hat{\mathbf{x}}\frac{\partial E_y}{\partial z} + \hat{\mathbf{y}}\frac{\partial E_x}{\partial z}$$

Assuming that $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$, the source-free Maxwell's equations become:

$$\begin{aligned}
\nabla\times\boldsymbol{E} &= -\mu\frac{\partial \boldsymbol{H}}{\partial t} & & \hat{\mathbf{z}}\times\frac{\partial \boldsymbol{E}}{\partial z} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}\\
\nabla\times\boldsymbol{H} &= \epsilon\frac{\partial \boldsymbol{E}}{\partial t} & & \hat{\mathbf{z}}\times\frac{\partial \boldsymbol{H}}{\partial z} = \epsilon\frac{\partial \boldsymbol{E}}{\partial t}\\
\nabla\cdot\boldsymbol{E} &= 0 & \Rightarrow\quad & \frac{\partial E_z}{\partial z} = 0\\
\nabla\cdot\boldsymbol{H} &= 0 & & \frac{\partial H_z}{\partial z} = 0
\end{aligned} \tag{2.1.1}$$

An immediate consequence of uniformity is that $\boldsymbol{E}$ and $\boldsymbol{H}$ do not have components along the z-direction, that is, $E_z = H_z = 0$. Taking the dot-product of Ampère's law with the unit vector $\hat{\mathbf{z}}$, and using the identity $\hat{\mathbf{z}}\cdot(\hat{\mathbf{z}}\times\boldsymbol{A}) = 0$, we have:

$$\hat{\mathbf{z}}\cdot\left(\hat{\mathbf{z}}\times\frac{\partial \boldsymbol{H}}{\partial z}\right) = \epsilon\,\hat{\mathbf{z}}\cdot\frac{\partial \boldsymbol{E}}{\partial t} = 0 \quad\Rightarrow\quad \frac{\partial E_z}{\partial t} = 0$$

†The shorthand notation ∂_x stands for $\dfrac{\partial}{\partial x}$.

Because also $\partial_z E_z = 0$, it follows that E_z must be a constant, independent of z, t. Excluding static solutions, we may take this constant to be zero. Similarly, we have $H_z = 0$. Thus, the fields have components only along the x, y directions:

$$\begin{aligned}
\boldsymbol{E}(z,t) &= \hat{\mathbf{x}}\,E_x(z,t) + \hat{\mathbf{y}}\,E_y(z,t)\\
\boldsymbol{H}(z,t) &= \hat{\mathbf{x}}\,H_x(z,t) + \hat{\mathbf{y}}\,H_y(z,t)
\end{aligned} \qquad \text{(transverse fields)} \tag{2.1.2}$$

These fields must satisfy Faraday's and Ampère's laws in Eqs. (2.1.1). We rewrite these equations in a more convenient form by replacing ϵ and μ by:

$$\epsilon = \frac{1}{\eta c}, \quad \mu = \frac{\eta}{c}, \qquad \text{where} \quad c = \frac{1}{\sqrt{\mu\epsilon}}, \quad \eta = \sqrt{\frac{\mu}{\epsilon}} \tag{2.1.3}$$

Thus, c, η are the speed of light and characteristic impedance of the propagation medium. Then, the first two of Eqs. (2.1.1) may be written in the equivalent forms:

$$\begin{aligned}
\hat{\mathbf{z}}\times\frac{\partial \boldsymbol{E}}{\partial z} &= -\frac{1}{c}\,\eta\frac{\partial \boldsymbol{H}}{\partial t}\\
\eta\,\hat{\mathbf{z}}\times\frac{\partial \boldsymbol{H}}{\partial z} &= \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}
\end{aligned} \tag{2.1.4}$$

The first may be solved for $\partial_z\boldsymbol{E}$ by crossing it with $\hat{\mathbf{z}}$. Using the BAC-CAB rule, and noting that $\boldsymbol{E}$ has no z-component, we have:

$$\left(\hat{\mathbf{z}}\times\frac{\partial \boldsymbol{E}}{\partial z}\right)\times\hat{\mathbf{z}} = \frac{\partial \boldsymbol{E}}{\partial z}(\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}) - \hat{\mathbf{z}}\left(\hat{\mathbf{z}}\cdot\frac{\partial \boldsymbol{E}}{\partial z}\right) = \frac{\partial \boldsymbol{E}}{\partial z}$$

where we used $\hat{\mathbf{z}}\cdot\partial_z\boldsymbol{E} = \partial_z E_z = 0$ and $\hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$. It follows that Eqs. (2.1.4) may be replaced by the equivalent system:

$$\boxed{\begin{aligned}
\frac{\partial \boldsymbol{E}}{\partial z} &= -\frac{1}{c}\frac{\partial}{\partial t}(\eta\boldsymbol{H}\times\hat{\mathbf{z}})\\
\frac{\partial}{\partial z}(\eta\boldsymbol{H}\times\hat{\mathbf{z}}) &= -\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}
\end{aligned}} \tag{2.1.5}$$

Now all the terms have the same dimension. Eqs. (2.1.5) imply that both $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the *one-dimensional wave equation*. Indeed, differentiating the first equation with respect to z and using the second, we have:

$$\frac{\partial^2\boldsymbol{E}}{\partial z^2} = -\frac{1}{c}\frac{\partial}{\partial t}\frac{\partial}{\partial z}(\eta\boldsymbol{H}\times\hat{\mathbf{z}}) = \frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} \qquad \text{or,}$$

$$\boxed{\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E}(z,t) = 0} \qquad \text{(wave equation)} \tag{2.1.6}$$

and similarly for $\boldsymbol{H}$. Rather than solving the wave equation, we prefer to work directly with the coupled system (2.1.5). The system can be decoupled by introducing the so-called *forward* and *backward* electric fields defined as the linear combinations:

$$\begin{aligned}
\boldsymbol{E}_+ &= \frac{1}{2}(\boldsymbol{E} + \eta\boldsymbol{H}\times\hat{\mathbf{z}})\\
\boldsymbol{E}_- &= \frac{1}{2}(\boldsymbol{E} - \eta\boldsymbol{H}\times\hat{\mathbf{z}})
\end{aligned} \qquad \text{(forward and backward fields)} \tag{2.1.7}$$

Component-wise, these are:

$$E_{x\pm} = \frac{1}{2}(E_x \pm \eta H_y)\,, \quad E_{y\pm} = \frac{1}{2}(E_y \mp \eta H_x) \tag{2.1.8}$$

We show next that $\boldsymbol{E}_+(z,t)$ corresponds to a forward-moving wave, that is, moving towards the positive z-direction, and $\boldsymbol{E}_-(z,t)$, to a backward-moving wave. Eqs. (2.1.7) can be inverted to express $\boldsymbol{E},\boldsymbol{H}$ in terms of $\boldsymbol{E}_+,\boldsymbol{E}_-$. Adding and subtracting them, and using the BAC-CAB rule and the orthogonality conditions $\hat{\mathbf{z}}\cdot\boldsymbol{E}_\pm = 0$, we obtain:

$$\begin{aligned} \boldsymbol{E}(z,t) &= \boldsymbol{E}_+(z,t) + \boldsymbol{E}_-(z,t) \\ \boldsymbol{H}(z,t) &= \frac{1}{\eta}\,\hat{\mathbf{z}}\times\big[\boldsymbol{E}_+(z,t) - \boldsymbol{E}_-(z,t)\big] \end{aligned} \tag{2.1.9}$$

In terms of the forward and backward fields $\boldsymbol{E}_\pm$, the system of Eqs. (2.1.5) *decouples* into two separate equations:

$$\boxed{\begin{aligned} \frac{\partial \boldsymbol{E}_+}{\partial z} &= -\frac{1}{c}\,\frac{\partial \boldsymbol{E}_+}{\partial t} \\ \frac{\partial \boldsymbol{E}_-}{\partial z} &= +\frac{1}{c}\,\frac{\partial \boldsymbol{E}_-}{\partial t} \end{aligned}} \tag{2.1.10}$$

Indeed, using Eqs. (2.1.5), we verify:

$$\frac{\partial}{\partial z}(\boldsymbol{E}\pm\eta\boldsymbol{H}\times\hat{\mathbf{z}}) = -\frac{1}{c}\,\frac{\partial}{\partial t}(\eta\boldsymbol{H}\times\hat{\mathbf{z}}) \mp \frac{1}{c}\,\frac{\partial \boldsymbol{E}}{\partial t} = \mp\frac{1}{c}\,\frac{\partial}{\partial t}(\boldsymbol{E}\pm\eta\boldsymbol{H}\times\hat{\mathbf{z}})$$

Eqs. (2.1.10) can be solved by noting that the forward field $\boldsymbol{E}_+(z,t)$ must depend on z,t only through the combination $z-ct$ (for a proof, see Problem 2.1.) If we set $\boldsymbol{E}_+(z,t) = \boldsymbol{F}(z-ct)$, where $\boldsymbol{F}(\zeta)$ is an arbitrary function of its argument $\zeta = z-ct$, then we will have:

$$\left.\begin{aligned} \frac{\partial \boldsymbol{E}_+}{\partial z} &= \frac{\partial}{\partial z}\boldsymbol{F}(z-ct) = \frac{\partial \zeta}{\partial z}\,\frac{\partial \boldsymbol{F}(\zeta)}{\partial \zeta} = \frac{\partial \boldsymbol{F}(\zeta)}{\partial \zeta} \\ \frac{\partial \boldsymbol{E}_+}{\partial t} &= \frac{\partial}{\partial t}\boldsymbol{F}(z-ct) = \frac{\partial \zeta}{\partial t}\,\frac{\partial \boldsymbol{F}(\zeta)}{\partial \zeta} = -c\,\frac{\partial \boldsymbol{F}(\zeta)}{\partial \zeta} \end{aligned}\right. \quad\Rightarrow\quad \frac{\partial \boldsymbol{E}_+}{\partial z} = -\frac{1}{c}\,\frac{\partial \boldsymbol{E}_+}{\partial t}$$

Vectorially, $\boldsymbol{F}$ must have only x,y components, $\boldsymbol{F} = \hat{\mathbf{x}}F_x + \hat{\mathbf{y}}F_y$, that is, it must be transverse to the propagation direction, $\hat{\mathbf{z}}\cdot\boldsymbol{F} = 0$.

Similarly, we find from the second of Eqs. (2.1.10) that $\boldsymbol{E}_-(z,t)$ must depend on z,t through the combination $z+ct$, so that $\boldsymbol{E}_-(z,t) = \boldsymbol{G}(z+ct)$, where $\boldsymbol{G}(\xi)$ is an arbitrary (transverse) function of $\xi = z+ct$. In conclusion, the most general solutions for the forward and backward fields of Eqs. (2.1.10) are:

$$\begin{aligned} \boldsymbol{E}_+(z,t) &= \boldsymbol{F}(z-ct) \\ \boldsymbol{E}_-(z,t) &= \boldsymbol{G}(z+ct) \end{aligned} \tag{2.1.11}$$

with arbitrary functions $\boldsymbol{F}$ and $\boldsymbol{G}$, such that $\hat{\mathbf{z}}\cdot\boldsymbol{F} = \hat{\mathbf{z}}\cdot\boldsymbol{G} = 0$.

Inserting these into the inverse formula (2.1.9), we obtain the most general solution of (2.1.5), expressed as a linear combination of forward and backward waves:

$$\boxed{\begin{aligned} \boldsymbol{E}(z,t) &= \boldsymbol{F}(z-ct) + \boldsymbol{G}(z+ct) \\ \boldsymbol{H}(z,t) &= \frac{1}{\eta}\,\hat{\mathbf{z}}\times\big[\boldsymbol{F}(z-ct) - \boldsymbol{G}(z+ct)\big] \end{aligned}} \tag{2.1.12}$$

The term $\boldsymbol{E}_+(z,t) = \boldsymbol{F}(z-ct)$ represents a wave propagating with speed c in the positive z-direction, while $\boldsymbol{E}_-(z,t) = \boldsymbol{G}(z+ct)$ represents a wave traveling in the negative z-direction.

To see this, consider the forward field at a later time $t+\Delta t$. During the time interval Δt, the wave moves in the positive z-direction by a distance $\Delta z = c\Delta t$. Indeed, we have:

$$\left.\begin{aligned} \boldsymbol{E}_+(z,t+\Delta t) &= \boldsymbol{F}\big(z-c(t+\Delta t)\big) = \boldsymbol{F}(z-c\Delta t-ct) \\ \boldsymbol{E}_+(z-\Delta z,t) &= \boldsymbol{F}\big((z-\Delta z)-ct\big) = \boldsymbol{F}(z-c\Delta t-ct) \end{aligned}\right. \;\Rightarrow\; \boldsymbol{E}_+(z,t+\Delta t) = \boldsymbol{E}_+(z-\Delta z,t)$$

This states that the forward field at time $t+\Delta t$ is the same as the field at time t, but translated to the *right* along the z-axis by a distance $\Delta z = c\Delta t$. Equivalently, the field at location $z+\Delta z$ at time t is the same as the field at location z at the earlier time $t-\Delta t = t-\Delta z/c$, that is,

$$\boldsymbol{E}_+(z+\Delta z,t) = \boldsymbol{E}_+(z,t-\Delta t)$$

Similarly, we find that $\boldsymbol{E}_-(z,t+\Delta t) = \boldsymbol{E}_-(z+\Delta z,t)$, which states that the backward field at time $t+\Delta t$ is the same as the field at time t, translated to the *left* by a distance Δz. Fig. 2.1.1 depicts these two cases.

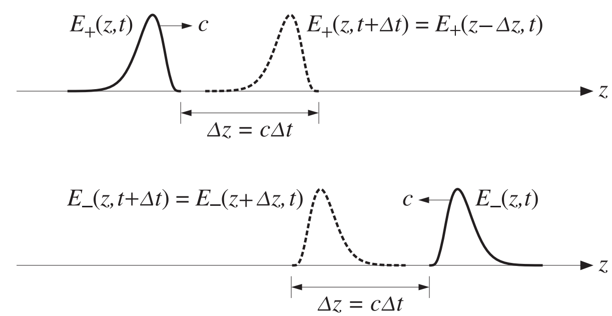
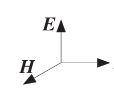


Fig. 2.1.1 Forward and backward waves.

The two special cases corresponding to forward waves only $(\boldsymbol{G}=0)$, or to backward ones $(\boldsymbol{F}=0)$, are of particular interest. For the forward case, we have:

$$\begin{aligned} \boldsymbol{E}(z,t) &= \boldsymbol{F}(z-ct) \\ \boldsymbol{H}(z,t) &= \frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{F}(z-ct) = \frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t) \end{aligned} \tag{2.1.13}$$



This solution has the following properties: (a) The field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to each other, $\boldsymbol{E}\cdot\boldsymbol{H}=0$, while they are transverse to the z-direction, (b) The three vectors $\{\boldsymbol{E},\boldsymbol{H},\hat{\mathbf{z}}\}$ form a *right-handed* vector system as shown in the figure, in the sense that $\boldsymbol{E}\times\boldsymbol{H}$ points in the direction of $\hat{\mathbf{z}}$, (c) The ratio of $\boldsymbol{E}$ to $\boldsymbol{H}\times\hat{\mathbf{z}}$ is independent of z,t and equals the characteristic impedance η of the propagation medium; indeed:

$$\boldsymbol{H}(z,t)=\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t)\quad\Rightarrow\quad\boldsymbol{E}(z,t)=\eta\boldsymbol{H}(z,t)\times\hat{\mathbf{z}}\tag{2.1.14}$$

The electromagnetic energy of such forward wave flows in the positive z-direction. With the help of the BAC-CAB rule, we find for the Poynting vector:

$$\boldsymbol{\mathcal{P}}=\boldsymbol{E}\times\boldsymbol{H}=\hat{\mathbf{z}}\,\frac{1}{\eta}|\boldsymbol{F}|^2=c\,\hat{\mathbf{z}}\,\epsilon|\boldsymbol{F}|^2\tag{2.1.15}$$

where we denoted $|\boldsymbol{F}|^2=\boldsymbol{F}\cdot\boldsymbol{F}$ and replaced $1/\eta=c\epsilon$. The electric and magnetic energy densities (per unit volume) turn out to be equal to each other. Because $\hat{\mathbf{z}}$ and $\boldsymbol{F}$ are mutually orthogonal, we have for the cross product $|\hat{\mathbf{z}}\times\boldsymbol{F}|=|\hat{\mathbf{z}}||\boldsymbol{F}|=|\boldsymbol{F}|$. Then,

$$\begin{aligned}w_e&=\frac{1}{2}\epsilon\,|\boldsymbol{E}|^2=\frac{1}{2}\epsilon|\boldsymbol{F}|^2\\ w_m&=\frac{1}{2}\mu\,|\boldsymbol{H}|^2=\frac{1}{2}\mu\,\frac{1}{\eta^2}|\hat{\mathbf{z}}\times\boldsymbol{F}|^2=\frac{1}{2}\epsilon\,|\boldsymbol{F}|^2=w_e\end{aligned}$$

where we replaced $\mu/\eta^2=\epsilon$. Thus, the total energy density of the forward wave will be:

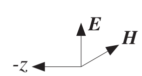
$$w=w_e+w_m=2w_e=\epsilon|\boldsymbol{F}|^2\tag{2.1.16}$$

In accordance with the flux/density relationship of Eq. (1.6.2), the transport velocity of the electromagnetic energy is found to be:

$$\boldsymbol{v}=\frac{\boldsymbol{\mathcal{P}}}{w}=\frac{c\,\hat{\mathbf{z}}\,\epsilon|\boldsymbol{F}|^2}{\epsilon|\boldsymbol{F}|^2}=c\,\hat{\mathbf{z}}$$

As expected, the energy of the forward-moving wave is being transported at a speed c along the positive z-direction. Similar results can be derived for the backward-moving solution that has $\boldsymbol{F}=0$ and $\boldsymbol{G}\neq 0$. The fields are now:

$$\begin{aligned}\boldsymbol{E}(z,t)&=\boldsymbol{G}(z+ct)\\ \boldsymbol{H}(z,t)&=-\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{G}(z+ct)=-\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t)\end{aligned}\tag{2.1.17}$$



The Poynting vector becomes $\boldsymbol{\mathcal{P}}=\boldsymbol{E}\times\boldsymbol{H}=-c\,\hat{\mathbf{z}}\,\epsilon|\boldsymbol{G}|^2$ and points in the negative z-direction, that is, the propagation direction. The energy transport velocity is $\boldsymbol{v}=-c\,\hat{\mathbf{z}}$. Now, the vectors $\{\boldsymbol{E},\boldsymbol{H},-\hat{\mathbf{z}}\}$ form a right-handed system, as shown. The ratio of E to H is still equal to η, provided we replace $\hat{\mathbf{z}}$ with $-\hat{\mathbf{z}}$:

$$\boldsymbol{H}(z,t)=\frac{1}{\eta}(-\hat{\mathbf{z}})\times\boldsymbol{E}(z,t)\quad\Rightarrow\quad\boldsymbol{E}(z,t)=\eta\,\boldsymbol{H}(z,t)\times(-\hat{\mathbf{z}})$$

In the general case of Eq. (2.1.12), the E/H ratio does not remain constant. The Poynting vector and energy density consist of a part due to the forward wave and a part due to the backward one:

$$\begin{aligned}\boldsymbol{\mathcal{P}}&=\boldsymbol{E}\times\boldsymbol{H}=c\,\hat{\mathbf{z}}\left(\epsilon|\boldsymbol{F}|^2-\epsilon|\boldsymbol{G}|^2\right)\\ w&=\frac{1}{2}\epsilon|\boldsymbol{E}|^2+\frac{1}{2}\mu|\boldsymbol{H}|^2=\epsilon|\boldsymbol{F}|^2+\epsilon|\boldsymbol{G}|^2\end{aligned}\tag{2.1.18}$$

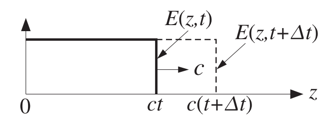
Example 2.1.1: A source located at $z=0$ generates an electric field $\boldsymbol{E}(0,t)=\hat{\mathbf{x}}E_0\,u(t)$, where $u(t)$ is the unit-step function, and E_0, a constant. The field is launched towards the positive z-direction. Determine expressions for $\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(z,t)$.

Solution: For a forward-moving wave, we have $\boldsymbol{E}(z,t)=\boldsymbol{F}(z-ct)=\boldsymbol{F}\big(0-c(t-z/c)\big)$, which implies that $\boldsymbol{E}(z,t)$ is completely determined by $\boldsymbol{E}(z,0)$, or alternatively, by $\boldsymbol{E}(0,t)$:

$$\boldsymbol{E}(z,t)=\boldsymbol{E}(z-ct,0)=\boldsymbol{E}(0,t-z/c)$$

Using this property, we find for the electric and magnetic fields:

$$\begin{aligned}\boldsymbol{E}(z,t)&=\boldsymbol{E}(0,t-z/c)=\hat{\mathbf{x}}E_0\,u(t-z/c)\\ \boldsymbol{H}(z,t)&=\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t)=\hat{\mathbf{y}}\,\frac{E_0}{\eta}\,u(t-z/c)\end{aligned}$$



Because of the unit-step, the non-zero values of the fields are restricted to $t-z/c\geq 0$, or, $z\leq ct$, that is, at time t the wavefront has propagated only up to position $z=ct$. The figure shows the expanding wavefronts at time t and $t+\Delta t$. □

Example 2.1.2: Consider the following three examples of electric fields specified at $t=0$, and describing forward or backward fields as indicated:

$$\begin{aligned}&\boldsymbol{E}(z,0)=\hat{\mathbf{x}}E_0\cos(kz) &&\text{(forward-moving)}\\ &\boldsymbol{E}(z,0)=\hat{\mathbf{y}}E_0\cos(kz) &&\text{(backward-moving)}\\ &\boldsymbol{E}(z,0)=\hat{\mathbf{x}}E_1\cos(k_1z)+\hat{\mathbf{y}}E_2\cos(k_2z) &&\text{(forward-moving)}\end{aligned}$$

where k,k_1,k_2 are given wavenumbers (measured in units of radians/m.) Determine the corresponding fields $\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(z,t)$.

Solution: For the forward-moving cases, we replace z by $z-ct$, and for the backward-moving case, by $z+ct$. We find in the three cases:

$$\begin{aligned}\boldsymbol{E}(z,t)&=\hat{\mathbf{x}}E_0\cos\big(k(z-ct)\big)=\hat{\mathbf{x}}E_0\cos(\omega t-kz)\\ \boldsymbol{E}(z,t)&=\hat{\mathbf{y}}E_0\cos\big(k(z+ct)\big)=\hat{\mathbf{y}}E_0\cos(\omega t+kz)\\ \boldsymbol{E}(z,t)&=\hat{\mathbf{x}}E_1\cos(\omega_1t-k_1z)+\hat{\mathbf{y}}E_2\cos(\omega_2t-k_2z)\end{aligned}$$

where $\omega=kc$, and $\omega_1=k_1c$, $\omega_2=k_2c$. The corresponding magnetic fields are:

$$\begin{aligned}\boldsymbol{H}(z,t)&=\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t)=\hat{\mathbf{y}}\,\frac{E_0}{\eta}\,\cos(\omega t-kz) &&\text{(forward)}\\ \boldsymbol{H}(z,t)&=-\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t)=\hat{\mathbf{x}}\,\frac{E_0}{\eta}\,\cos(\omega t+kz) &&\text{(backward)}\\ \boldsymbol{H}(z,t)&=\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}(z,t)=\hat{\mathbf{y}}\,\frac{E_1}{\eta}\,\cos(\omega_1t-k_1z)-\hat{\mathbf{x}}\,\frac{E_2}{\eta}\,\cos(\omega_2t-k_2z)\end{aligned}$$

The first two cases are single-frequency waves, and are discussed in more detail in the next section. The third case is a linear superposition of two waves with two different frequencies and polarizations. □

2.2 Monochromatic Waves

Uniform, single-frequency, plane waves propagating in a lossless medium are obtained as a special case of the previous section by assuming the harmonic time-dependence:

$$\begin{aligned} \boldsymbol{E}(x,y,z,t) &= \boldsymbol{E}(z)\,e^{j\omega t} \\ \boldsymbol{H}(x,y,z,t) &= \boldsymbol{H}(z)\,e^{j\omega t} \end{aligned} \tag{2.2.1}$$

where $\boldsymbol{E}(z)$ and $\boldsymbol{H}(z)$ are transverse with respect to the z-direction.

Maxwell's equations (2.1.5), or those of the decoupled system (2.1.10), may be solved very easily by replacing time derivatives by $\partial_t \to j\omega$. Then, Eqs. (2.1.10) become the first-order differential equations (see also Problem 2.3):

$$\frac{\partial \boldsymbol{E}_\pm(z)}{\partial z} = \mp jk\,\boldsymbol{E}_\pm(z)\,, \qquad \text{where} \quad k = \frac{\omega}{c} = \omega\sqrt{\mu\epsilon} \tag{2.2.2}$$

with solutions:

$$\begin{aligned} \boldsymbol{E}_+(z) &= \boldsymbol{E}_{0+}e^{-jkz} \qquad \text{(forward)} \\ \boldsymbol{E}_-(z) &= \boldsymbol{E}_{0-}e^{jkz} \qquad \text{(backward)} \end{aligned} \tag{2.2.3}$$

where $\boldsymbol{E}_{0\pm}$ are arbitrary (complex-valued) constant vectors such that $\hat{\mathbf{z}}\cdot\boldsymbol{E}_{0\pm} = 0$. The corresponding magnetic fields are:

$$\begin{aligned} \boldsymbol{H}_+(z) &= \frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}_+(z) = \frac{1}{\eta}\,(\hat{\mathbf{z}}\times\boldsymbol{E}_{0+})\,e^{-jkz} = \boldsymbol{H}_{0+}e^{-jkz} \\ \boldsymbol{H}_-(z) &= -\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}_-(z) = -\frac{1}{\eta}\,(\hat{\mathbf{z}}\times\boldsymbol{E}_{0-})\,e^{jkz} = \boldsymbol{H}_{0-}e^{jkz} \end{aligned} \tag{2.2.4}$$

where we defined the constant amplitudes of the magnetic fields:

$$\boldsymbol{H}_{0\pm} = \pm\frac{1}{\eta}\,\hat{\mathbf{z}}\times\boldsymbol{E}_{0\pm} \tag{2.2.5}$$

Inserting (2.2.3) into (2.1.9), we obtain the general solution for single-frequency waves, expressed as a superposition of forward and backward components:

$$\boxed{\begin{aligned} \boldsymbol{E}(z) &= \boldsymbol{E}_{0+}e^{-jkz} + \boldsymbol{E}_{0-}e^{jkz} \\ \boldsymbol{H}(z) &= \frac{1}{\eta}\,\hat{\mathbf{z}}\times\big[\boldsymbol{E}_{0+}e^{-jkz} - \boldsymbol{E}_{0-}e^{jkz}\big] \end{aligned}} \qquad \text{(forward + backward waves)} \tag{2.2.6}$$

Setting $\boldsymbol{E}_{0\pm} = \hat{\mathbf{x}}A_\pm + \hat{\mathbf{y}}B_\pm$, and noting that $\hat{\mathbf{z}}\times\boldsymbol{E}_{0\pm} = \hat{\mathbf{z}}\times(\hat{\mathbf{x}}A_\pm + \hat{\mathbf{y}}B_\pm) = \hat{\mathbf{y}}A_\pm - \hat{\mathbf{x}}B_\pm$, we may rewrite (2.2.6) in terms of its cartesian components:

$$\begin{aligned} &E_x(z) = A_+e^{-jkz} + A_-e^{jkz}\,, && E_y(z) = B_+e^{-jkz} + B_-e^{jkz} \\ &H_y(z) = \frac{1}{\eta}\big[A_+e^{-jkz} - A_-e^{jkz}\big]\,, && H_x(z) = -\frac{1}{\eta}\big[B_+e^{-jkz} - B_-e^{jkz}\big] \end{aligned} \tag{2.2.7}$$

Wavefronts are defined, in general, to be the surfaces of constant phase. A forward moving wave $\boldsymbol{E}(z) = \boldsymbol{E}_0 e^{-jkz}$ corresponds to the time-varying field:

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_0 e^{j\omega t - jkz} = \boldsymbol{E}_0 e^{-j\varphi(z,t)}\,, \qquad \text{where} \quad \varphi(z,t) = kz - \omega t$$

A surface of constant phase is obtained by setting $\varphi(z,t) =$ const. Denoting this constant by $\phi_0 = kz_0$ and using the property $c = \omega/k$, we obtain the condition:

$$\varphi(z,t) = \varphi_0 \quad\Rightarrow\quad kz - \omega t = kz_0 \quad\Rightarrow\quad z = ct + z_0$$

Thus, the wavefront is the xy-plane intersecting the z-axis at the point $z = ct + z_0$, moving forward with velocity c. This justifies the term "plane wave."

A backward-moving wave will have planar wavefronts parametrized by $z = -ct + z_0$, that is, moving backwards. A wave that is a linear combination of forward and backward components, may be thought of as having two planar wavefronts, one moving forward, and the other backward.

The relationships (2.2.5) imply that the vectors $\{\boldsymbol{E}_{0+}, \boldsymbol{H}_{0+}, \hat{\mathbf{z}}\}$ and $\{\boldsymbol{E}_{0-}, \boldsymbol{H}_{0-}, -\hat{\mathbf{z}}\}$ will form right-handed orthogonal systems. The magnetic field $\boldsymbol{H}_{0\pm}$ is perpendicular to the electric field $\boldsymbol{E}_{0\pm}$ and the cross-product $\boldsymbol{E}_{0\pm}\times\boldsymbol{H}_{0\pm}$ points towards the direction of propagation, that is, $\pm\hat{\mathbf{z}}$. Fig. 2.2.1 depicts the case of a forward propagating wave.

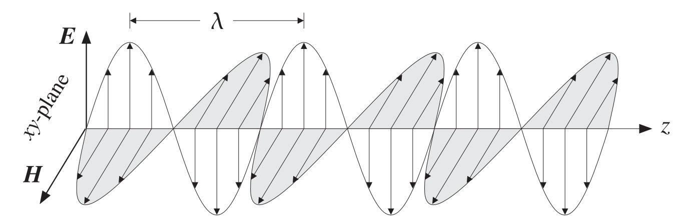


Fig. 2.2.1 Forward uniform plane wave.

The *wavelength* λ is the distance by which the phase of the sinusoidal wave changes by 2π radians. Since the propagation factor e^{-jkz} accumulates a phase of k radians per meter, we have by definition that $k\lambda = 2\pi$. The wavelength λ can be expressed via the frequency of the wave in Hertz, $f = \omega/2\pi$, as follows:

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}} \tag{2.2.8}$$

If the propagation medium is free space, we use the vacuum values of the parameters $\{\epsilon, \mu, c, \eta\}$, that is, $\{\epsilon_0, \mu_0, c_0, \eta_0\}$. The free-space wavelength and corresponding wavenumber are:

$$\lambda_0 = \frac{2\pi}{k_0} = \frac{c_0}{f}\,, \quad k_0 = \frac{\omega}{c_0} \tag{2.2.9}$$

In a lossless but non-magnetic ($\mu = \mu_0$) dielectric with refractive index $n = \sqrt{\epsilon/\epsilon_0}$, the speed of light c, wavelength λ, and characteristic impedance η are all reduced by a scale factor n compared to the free-space values, whereas the wavenumber k is increased by a factor of n. Indeed, using the definitions $c = 1/\sqrt{\mu_0\epsilon}$ and $\eta = \sqrt{\mu_0/\epsilon}$, we have:

$$\boxed{c = \frac{c_0}{n}\,, \quad \eta = \frac{\eta_0}{n}\,, \quad \lambda = \frac{\lambda_0}{n}\,, \quad k = nk_0} \tag{2.2.10}$$

Example 2.2.1: A microwave transmitter operating at the carrier frequency of 6 GHz is protected by a Plexiglas radome whose permittivity is $\epsilon = 3\epsilon_0$.

The refractive index of the radome is $n = \sqrt{\epsilon/\epsilon_0} = \sqrt{3} = 1.73$. The free-space wavelength and the wavelength inside the radome material are:

$$\lambda_0 = \frac{c_0}{f} = \frac{3\times 10^8}{6\times 10^9} = 0.05\text{ m} = 5\text{ cm}, \qquad \lambda = \frac{\lambda_0}{n} = \frac{5}{1.73} = 2.9\text{ cm}$$

We will see later that if the radome is to be transparent to the wave, its thickness must be chosen to be equal to one-half wavelength, $l = \lambda/2$. Thus, $l = 2.9/2 = 1.45$ cm. □

Example 2.2.2: The nominal speed of light in vacuum is $c_0 = 3\times 10^8$ m/s. Because of the relationship $c_0 = \lambda f$, it may be expressed in the following suggestive units that are appropriate in different application contexts:

$$\begin{aligned}
c_0 \;=\;& 5000\text{ km}\times 60\text{ Hz} && \text{(power systems)}\\
& 300\text{ m}\times 1\text{ MHz} && \text{(AM radio)}\\
& 40\text{ m}\times 7.5\text{ MHz} && \text{(amateur radio)}\\
& 3\text{ m}\times 100\text{ MHz} && \text{(FM radio, TV)}\\
& 30\text{ cm}\times 1\text{ GHz} && \text{(cell phones)}\\
& 10\text{ cm}\times 3\text{ GHz} && \text{(waveguides, radar)}\\
& 3\text{ cm}\times 10\text{ GHz} && \text{(radar, satellites)}\\
& 1.5\ \mu\text{m}\times 200\text{ THz} && \text{(optical fibers)}\\
& 500\text{ nm}\times 600\text{ THz} && \text{(visible spectrum)}\\
& 100\text{ nm}\times 3000\text{ THz} && \text{(UV)}
\end{aligned}$$

Similarly, in terms of length/time of propagation:

$$\begin{aligned}
c_0 \;=\;& 36\,000\text{ km}/120\text{ msec} && \text{(geosynchronous satellites)}\\
& 300\text{ km/msec} && \text{(power lines)}\\
& 300\text{ m}/\mu\text{sec} && \text{(transmission lines)}\\
& 30\text{ cm/nsec} && \text{(circuit boards)}
\end{aligned}$$

The typical half-wave monopole antenna (half of a half-wave dipole over a ground plane) has length $\lambda/4$ and is used in many applications, such as AM, FM, and cell phones. Thus, one can predict that the lengths of AM radio, FM radio, and cell phone antennas will be of the order of 75 m, 0.75 m, and 7.5 cm, respectively.

A more detailed list of electromagnetic frequency bands is given in Appendix B. The precise value of c_0 and the values of other physical constants are given in Appendix A. □

Wave propagation effects become important, and cannot be ignored, whenever the physical length of propagation is comparable to the wavelength λ. It follows from Eqs. (2.2.2) that the incremental change of a forward-moving electric field in propagating from z to $z + \Delta z$ is:

$$\frac{|\Delta \boldsymbol{E}_+|}{|\boldsymbol{E}_+|} = k\Delta z = 2\pi\,\frac{\Delta z}{\lambda} \tag{2.2.11}$$

Thus, the change in the electric field can be ignored only if $\Delta z \ll \lambda$, otherwise, propagation effects must be taken into account.

For example, for an integrated circuit operating at 10 GHz, we have $\lambda = 3$ cm, which is comparable to the physical dimensions of the circuit.

Similarly, a cellular base station antenna is connected to the transmitter circuits by several meters of coaxial cable. For a 1-GHz system, the wavelength is 0.3 m, which implies that a 30-meter cable will be equivalent to 100 wavelengths.

2.3 Energy Density and Flux

The time-averaged energy density and flux of a uniform plane wave can be determined by Eq. (1.9.6). As in the previous section, the energy is shared equally by the electric and magnetic fields (in the forward or backward cases.) This is a general result for most wave propagation and waveguide problems.

The energy flux will be in the direction of propagation. For either a forward- or a backward-moving wave, we have from Eqs. (1.9.6) and (2.2.5):

$$\begin{aligned}
w_e &= \frac{1}{2}\,\mathrm{Re}\Big[\frac{1}{2}\epsilon\,\boldsymbol{E}_\pm(z)\cdot\boldsymbol{E}_\pm^*(z)\Big] = \frac{1}{2}\,\mathrm{Re}\Big[\frac{1}{2}\epsilon\,\boldsymbol{E}_{0\pm}e^{-jkz}\cdot\boldsymbol{E}_{0\pm}^*e^{jkz}\Big] = \frac{1}{4}\epsilon|\boldsymbol{E}_{0\pm}|^2\\
w_m &= \frac{1}{2}\,\mathrm{Re}\Big[\frac{1}{2}\mu\,\boldsymbol{H}_\pm(z)\cdot\boldsymbol{H}_\pm^*(z)\Big] = \frac{1}{4}\mu|\boldsymbol{H}_{0\pm}|^2 = \frac{1}{4}\mu\frac{1}{\eta^2}|\hat{\mathbf{z}}\times\boldsymbol{E}_{0\pm}|^2 = \frac{1}{4}\epsilon|\boldsymbol{E}_{0\pm}|^2 = w_e
\end{aligned}$$

Thus, the electric and magnetic energy densities are equal and the total density is:

$$w = w_e + w_m = 2w_e = \frac{1}{2}\epsilon|\boldsymbol{E}_{0\pm}|^2 \tag{2.3.1}$$

For the time-averaged Poynting vector, we have similarly:

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\,\mathrm{Re}\big[\,\boldsymbol{E}_\pm(z)\times\boldsymbol{H}_\pm^*(z)\,\big] = \frac{1}{2\eta}\,\mathrm{Re}\big[\,\boldsymbol{E}_{0\pm}\times(\pm\hat{\mathbf{z}}\times\boldsymbol{E}_{0\pm}^*)\,\big]$$

Using the BAC-CAB rule and the orthogonality property $\hat{\mathbf{z}}\cdot\boldsymbol{E}_{0\pm} = 0$, we find:

$$\boldsymbol{\mathcal{P}} = \pm\hat{\mathbf{z}}\,\frac{1}{2\eta}|\boldsymbol{E}_{0\pm}|^2 = \pm c\,\hat{\mathbf{z}}\,\frac{1}{2}\,\epsilon|\boldsymbol{E}_{0\pm}|^2 \tag{2.3.2}$$

Thus, the energy flux is in the direction of propagation, that is, $\pm\hat{\mathbf{z}}$. The corresponding energy velocity is, as in the previous section:

$$\mathbf{v} = \frac{\boldsymbol{\mathcal{P}}}{w} = \pm c\,\hat{\mathbf{z}} \tag{2.3.3}$$

In the more general case of forward and backward waves, we find:

$$\begin{aligned}
w &= \frac{1}{4}\,\mathrm{Re}\big[\epsilon\,\boldsymbol{E}(z)\cdot\boldsymbol{E}^*(z) + \mu\,\boldsymbol{H}(z)\cdot\boldsymbol{H}^*(z)\big] = \frac{1}{2}\epsilon|\boldsymbol{E}_{0+}|^2 + \frac{1}{2}\epsilon|\boldsymbol{E}_{0-}|^2\\
\boldsymbol{\mathcal{P}} &= \frac{1}{2}\,\mathrm{Re}\big[\boldsymbol{E}(z)\times\boldsymbol{H}^*(z)\big] = \hat{\mathbf{z}}\left(\frac{1}{2\eta}|\boldsymbol{E}_{0+}|^2 - \frac{1}{2\eta}|\boldsymbol{E}_{0-}|^2\right)
\end{aligned} \tag{2.3.4}$$

Thus, the total energy is the sum of the energies of the forward and backward components, whereas the net energy flux (to the right) is the difference between the forward and backward fluxes.

2.4 Wave Impedance

For forward or backward fields, the ratio of $\boldsymbol{E}(z)$ to $\boldsymbol{H}(z)\times\hat{\mathbf{z}}$ is constant and equal to the characteristic impedance of the medium. Indeed, it follows from Eq. (2.2.4) that

$$\boldsymbol{E}_{\pm}(z)=\pm\eta\boldsymbol{H}_{\pm}(z)\times\hat{\mathbf{z}}$$

However, this property is not true for the more general solution given by Eqs. (2.2.6). In general, the ratio of $\boldsymbol{E}(z)$ to $\boldsymbol{H}(z)\times\hat{\mathbf{z}}$ is called the *wave impedance*. Because of the vectorial character of the fields, we must define the ratio in terms of the corresponding x- and y-components:

$$\begin{aligned} Z_x(z) &= \frac{\big[\boldsymbol{E}(z)\big]_x}{\big[\boldsymbol{H}(z)\times\hat{\mathbf{z}}\big]_x}=\frac{E_x(z)}{H_y(z)} \\ Z_y(z) &= \frac{\big[\boldsymbol{E}(z)\big]_y}{\big[\boldsymbol{H}(z)\times\hat{\mathbf{z}}\big]_y}=-\frac{E_y(z)}{H_x(z)} \end{aligned}\qquad\text{(wave impedances)}\qquad(2.4.1)$$

Using the cartesian expressions of Eq. (2.2.7), we find:

$$\begin{aligned} Z_x(z) &= \frac{E_x(z)}{H_y(z)}=\eta\,\frac{A_+e^{-jkz}+A_-e^{jkz}}{A_+e^{-jkz}-A_-e^{jkz}} \\ Z_y(z) &= -\frac{E_y(z)}{H_x(z)}=\eta\,\frac{B_+e^{-jkz}+B_-e^{jkz}}{B_+e^{-jkz}-B_-e^{jkz}} \end{aligned}\qquad\text{(wave impedances)}\qquad(2.4.2)$$

Thus, the wave impedances are nontrivial functions of z. For forward waves (that is, with $A_-=B_-=0$), we have $Z_x(z)=Z_y(z)=\eta$. For backward waves ($A_+=B_+=0$), we have $Z_x(z)=Z_y(z)=-\eta$.

The wave impedance is a very useful concept in the subject of multiple dielectric interfaces and the matching of transmission lines. We will explore its use later on.

2.5 Polarization

Consider a forward-moving wave and let $\boldsymbol{E}_0=\hat{\mathbf{x}}A_++\hat{\mathbf{y}}B_+$ be its complex-valued phasor amplitude, so that $\boldsymbol{E}(z)=\boldsymbol{E}_0e^{-jkz}=(\hat{\mathbf{x}}A_++\hat{\mathbf{y}}B_+)e^{-jkz}$. The time-varying field is obtained by restoring the factor $e^{j\omega t}$:

$$\boldsymbol{E}(z,t)=(\hat{\mathbf{x}}A_++\hat{\mathbf{y}}B_+)e^{j\omega t-jkz}$$

The polarization of a plane wave is defined to be the direction of the electric field. For example, if $B_+=0$, the E-field is along the x-direction and the wave will be linearly polarized.

More precisely, polarization is the direction of the time-varying real-valued field $\boldsymbol{\mathcal{E}}(z,t)=\text{Re}\big[\boldsymbol{E}(z,t)\big]$. At any fixed point z, the vector $\boldsymbol{\mathcal{E}}(z,t)$ may be along a fixed linear direction or it may be rotating as a function of t, tracing a circle or an ellipse.

The polarization properties of the plane wave are determined by the relative magnitudes and phases of the complex-valued constants A_+,B_+. Writing them in their polar forms $A_+=Ae^{j\phi_a}$ and $B_+=Be^{j\phi_b}$, where A,B are *positive magnitudes*, we obtain:

$$\boldsymbol{E}(z,t)=\big(\hat{\mathbf{x}}Ae^{j\phi_a}+\hat{\mathbf{y}}Be^{j\phi_b}\big)e^{j\omega t-jkz}=\hat{\mathbf{x}}Ae^{j(\omega t-kz+\phi_a)}+\hat{\mathbf{y}}Be^{j(\omega t-kz+\phi_b)}\qquad(2.5.1)$$

Extracting real parts and setting $\boldsymbol{\mathcal{E}}(z,t)=\text{Re}\big[\boldsymbol{E}(z,t)\big]=\hat{\mathbf{x}}\mathcal{E}_x(z,t)+\hat{\mathbf{y}}\mathcal{E}_y(z,t)$, we find the corresponding real-valued x,y components:

$$\begin{aligned} \mathcal{E}_x(z,t) &= A\cos(\omega t-kz+\phi_a) \\ \mathcal{E}_y(z,t) &= B\cos(\omega t-kz+\phi_b) \end{aligned}\qquad(2.5.2)$$

For a backward moving field, we replace k by $-k$ in the same expression. To determine the polarization of the wave, we consider the time-dependence of these fields at some *fixed* point along the z-axis, say at $z=0$:

$$\begin{aligned} \mathcal{E}_x(t) &= A\cos(\omega t+\phi_a) \\ \mathcal{E}_y(t) &= B\cos(\omega t+\phi_b) \end{aligned}\qquad(2.5.3)$$

The electric field vector $\boldsymbol{\mathcal{E}}(t)=\hat{\mathbf{x}}\mathcal{E}_x(t)+\hat{\mathbf{y}}\mathcal{E}_y(t)$ will be rotating on the xy-plane with angular frequency ω, with its tip tracing, in general, an ellipse. To see this, we expand Eq. (2.5.3) using a trigonometric identity:

$$\begin{aligned} \mathcal{E}_x(t) &= A\big[\cos\omega t\cos\phi_a-\sin\omega t\sin\phi_a\big] \\ \mathcal{E}_y(t) &= B\big[\cos\omega t\cos\phi_b-\sin\omega t\sin\phi_b\big] \end{aligned}$$

Solving for $\cos\omega t$ and $\sin\omega t$ in terms of $\mathcal{E}_x(t),\mathcal{E}_y(t)$, we find:

$$\begin{aligned} \cos\omega t\sin\phi &= \frac{\mathcal{E}_y(t)}{B}\sin\phi_a-\frac{\mathcal{E}_x(t)}{A}\sin\phi_b \\ \sin\omega t\sin\phi &= \frac{\mathcal{E}_y(t)}{B}\cos\phi_a-\frac{\mathcal{E}_x(t)}{A}\cos\phi_b \end{aligned}$$

where we defined the relative phase angle $\phi=\phi_a-\phi_b$.

Forming the sum of the squares of the two equations and using the trigonometric identity $\sin^2\omega t+\cos^2\omega t=1$, we obtain a quadratic equation for the components $\mathcal{E}_x$ and $\mathcal{E}_y$, which describes an ellipse on the $\mathcal{E}_x,\mathcal{E}_y$ plane:

$$\left(\frac{\mathcal{E}_y(t)}{B}\sin\phi_a-\frac{\mathcal{E}_x(t)}{A}\sin\phi_b\right)^2+\left(\frac{\mathcal{E}_y(t)}{B}\cos\phi_a-\frac{\mathcal{E}_x(t)}{A}\cos\phi_b\right)^2=\sin^2\phi$$

This simplifies into:

$$\boxed{\frac{\mathcal{E}_x^2}{A^2}+\frac{\mathcal{E}_y^2}{B^2}-2\cos\phi\,\frac{\mathcal{E}_x\mathcal{E}_y}{AB}=\sin^2\phi}\qquad\text{(polarization ellipse)}\qquad(2.5.4)$$

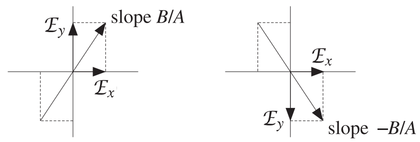
Depending on the values of the three quantities $\{A,B,\phi\}$ this *polarization ellipse* may be an ellipse, a circle, or a straight line. The electric field is accordingly called elliptically, circularly, or linearly polarized.

To get *linear polarization*, we set $\phi = 0$ or $\phi = \pi$, corresponding to $\phi_a = \phi_b = 0$, or $\phi_a = 0, \phi_b = -\pi$, so that the phasor amplitudes are $\mathbf{E}_0 = \hat{\mathbf{x}}\,A \pm \hat{\mathbf{y}}\,B$. Then, Eq. (2.5.4) degenerates into:

$$\frac{\mathcal{E}_x^2}{A^2} + \frac{\mathcal{E}_y^2}{B^2} \mp 2\,\frac{\mathcal{E}_x\mathcal{E}_y}{AB} = 0 \qquad \Rightarrow \qquad \left(\frac{\mathcal{E}_x}{A} \mp \frac{\mathcal{E}_y}{B}\right)^2 = 0$$

representing the straight lines:

$$\mathcal{E}_y = \pm\frac{B}{A}\,\mathcal{E}_x$$



The fields (2.5.2) take the forms, in the two cases $\phi = 0$ and $\phi = \pi$:

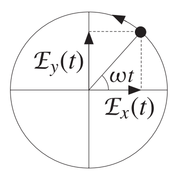
$$\begin{aligned} \mathcal{E}_x(t) &= A\cos\omega t \\ \mathcal{E}_y(t) &= B\cos\omega t \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{E}_x(t) &= A\cos\omega t \\ \mathcal{E}_y(t) &= B\cos(\omega t - \pi) = -B\cos\omega t \end{aligned}$$

To get *circular polarization*, we set $A = B$ and $\phi = \pm\pi/2$. In this case, the polarization ellipse becomes the equation of a circle:

$$\frac{\mathcal{E}_x^2}{A^2} + \frac{\mathcal{E}_y^2}{A^2} = 1$$

The *sense* of rotation, in conjunction with the direction of propagation, defines left-circular versus right-circular polarization. For the case, $\phi_a = 0$ and $\phi_b = -\pi/2$, we have $\phi = \phi_a - \phi_b = \pi/2$ and complex amplitude $\mathbf{E}_0 = A(\hat{\mathbf{x}} - j\hat{\mathbf{y}})$. Then,

$$\begin{aligned} \mathcal{E}_x(t) &= A\cos\omega t \\ \mathcal{E}_y(t) &= A\cos(\omega t - \pi/2) = A\sin\omega t \end{aligned}$$



Thus, the tip of the electric field vector rotates *counterclockwise* on the xy-plane. To decide whether this represents right or left circular polarization, we use the *IEEE convention*[115], which is as follows.

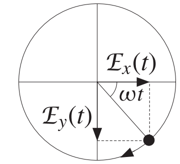
Curl the fingers of your left and right hands into a fist and point both thumbs towards the direction of propagation. If the fingers of your right (left) hand are curling in the direction of rotation of the electric field, then the polarization is right (left) polarized.†

Thus, in the present example, because we had a forward-moving field and the field is turning counterclockwise, the polarization will be right-circular. If the field were moving backwards, then it would be left-circular. For the case, $\phi = -\pi/2$, arising from $\phi_a = 0$

†Most engineering texts use the IEEE convention and most physics texts, the opposite convention.

and $\phi_b = \pi/2$, we have complex amplitude $\mathbf{E}_0 = A(\hat{\mathbf{x}} + j\hat{\mathbf{y}})$. Then, Eq. (2.5.3) becomes:

$$\begin{aligned} \mathcal{E}_x(t) &= A\cos\omega t \\ \mathcal{E}_y(t) &= A\cos(\omega t + \pi/2) = -A\sin\omega t \end{aligned}$$



The tip of the electric field vector rotates *clockwise* on the xy-plane. Since the wave is moving forward, this will represent left-circular polarization. Fig. 2.5.1 depicts the four cases of left/right polarization with forward/backward waves.

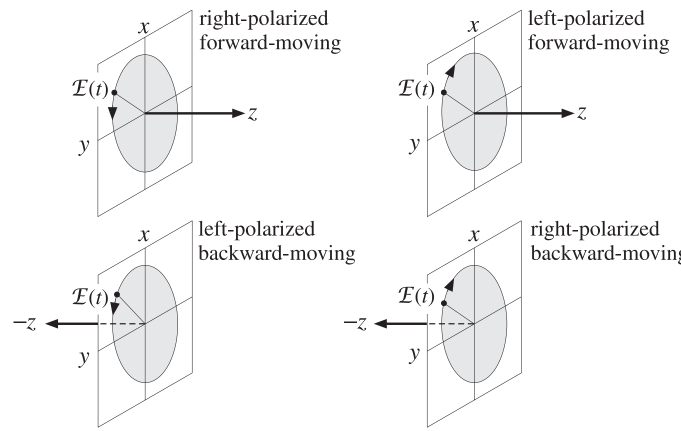


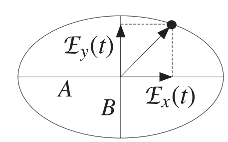
Fig. 2.5.1 Left and right circular polarizations.

To summarize, the electric field of a *circularly polarized* uniform plane wave will be, in its phasor form:

$$\begin{aligned} \mathbf{E}(z) &= A(\hat{\mathbf{x}} - j\hat{\mathbf{y}})\,e^{-jkz} && \text{(right-polarized, forward-moving)} \\ \mathbf{E}(z) &= A(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\,e^{-jkz} && \text{(left-polarized, forward-moving)} \\ \mathbf{E}(z) &= A(\hat{\mathbf{x}} - j\hat{\mathbf{y}})\,e^{jkz} && \text{(left-polarized, backward-moving)} \\ \mathbf{E}(z) &= A(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\,e^{jkz} && \text{(right-polarized, backward-moving)} \end{aligned}$$

If $A \neq B$, but the phase difference is still $\phi = \pm\pi/2$, we get an ellipse with major and minor axes oriented along the x, y directions. Eq. (2.5.4) will be now:

$$\frac{\mathcal{E}_x^2}{A^2} + \frac{\mathcal{E}_y^2}{B^2} = 1$$



Finally, if $A \neq B$ and ϕ is arbitrary, then the major/minor axes of the ellipse (2.5.4) will be rotated relative to the x, y directions. Fig. 2.5.2 illustrates the general case.

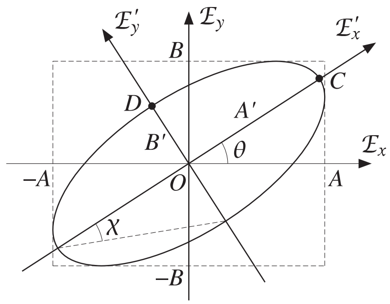


Fig. 2.5.2 General polarization ellipse.

It can be shown (see Problem 2.15) that the tilt angle θ is given by:

$$\tan 2\theta = \frac{2AB}{A^2 - B^2}\cos\phi \tag{2.5.5}$$

The ellipse semi-axes A', B', that is, the lengths OC and OD, are given by:

$$\begin{aligned} A' &= \sqrt{\frac{1}{2}(A^2+B^2) + \frac{s}{2}\sqrt{(A^2-B^2)^2 + 4A^2B^2\cos^2\phi}} \\ B' &= \sqrt{\frac{1}{2}(A^2+B^2) - \frac{s}{2}\sqrt{(A^2-B^2)^2 + 4A^2B^2\cos^2\phi}} \end{aligned} \tag{2.5.6}$$

where $s = \text{sign}(A - B)$. These results are obtained by defining the rotated coordinate system of the ellipse axes:

$$\begin{aligned} \mathcal{E}'_x &= \mathcal{E}_x\cos\theta + \mathcal{E}_y\sin\theta \\ \mathcal{E}'_y &= \mathcal{E}_y\cos\theta - \mathcal{E}_x\sin\theta \end{aligned} \tag{2.5.7}$$

and showing that Eq. (2.5.4) transforms into the standardized form:

$$\frac{\mathcal{E}'^2_x}{A'^2} + \frac{\mathcal{E}'^2_y}{B'^2} = 1 \tag{2.5.8}$$

The polarization ellipse is bounded by the rectangle with sides at the end-points $\pm A, \pm B$, as shown in the figure. To decide whether the elliptic polarization is left- or right-handed, we may use the same rules depicted in Fig. 2.5.1.

The angle χ subtended by the major to minor ellipse axes shown in Fig. 2.5.2 is given as follows and is discussed further in Problem 2.15:

$$\sin 2\chi = \frac{2AB}{A^2 + B^2}\,|\sin\phi|\,, \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4} \tag{2.5.9}$$

that is, it can be shown that $\tan\chi = B'/A'$ or A'/B', whichever is less than one.

Example 2.5.1: Determine the real-valued electric and magnetic field components and the polarization of the following fields specified in their phasor form (given in units of V/m):

a. $\mathbf{E}(z) = -3j\,\hat{\mathbf{x}}\,e^{-jkz}$
b. $\mathbf{E}(z) = \big(3\,\hat{\mathbf{x}} + 4\,\hat{\mathbf{y}}\big)e^{+jkz}$
c. $\mathbf{E}(z) = \big(-4\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}}\big)e^{-jkz}$
d. $\mathbf{E}(z) = \big(3e^{j\pi/3}\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}}\big)e^{+jkz}$
e. $\mathbf{E}(z) = \big(4\,\hat{\mathbf{x}} + 3e^{-j\pi/4}\,\hat{\mathbf{y}}\big)e^{-jkz}$
f. $\mathbf{E}(z) = \big(3e^{-j\pi/8}\,\hat{\mathbf{x}} + 4e^{j\pi/8}\,\hat{\mathbf{y}}\big)e^{+jkz}$
g. $\mathbf{E}(z) = \big(4e^{j\pi/4}\,\hat{\mathbf{x}} + 3e^{-j\pi/2}\,\hat{\mathbf{y}}\big)e^{-jkz}$
h. $\mathbf{E}(z) = \big(3e^{-j\pi/2}\,\hat{\mathbf{x}} + 4e^{j\pi/4}\,\hat{\mathbf{y}}\big)e^{+jkz}$

Solution: Restoring the $e^{j\omega t}$ factor and taking real-parts, we find the x, y electric field components, according to Eq. (2.5.2):

a. $\mathcal{E}_x(z,t) = 3\cos(\omega t - kz - \pi/2)$, $\mathcal{E}_y(z,t) = 0$
b. $\mathcal{E}_x(z,t) = 3\cos(\omega t + kz)$, $\mathcal{E}_y(z,t) = 4\cos(\omega t + kz)$
c. $\mathcal{E}_x(z,t) = 4\cos(\omega t - kz + \pi)$, $\mathcal{E}_y(z,t) = 3\cos(\omega t - kz)$
d. $\mathcal{E}_x(z,t) = 3\cos(\omega t + kz + \pi/3)$, $\mathcal{E}_y(z,t) = 3\cos(\omega t + kz)$
e. $\mathcal{E}_x(z,t) = 4\cos(\omega t - kz)$, $\mathcal{E}_y(z,t) = 3\cos(\omega t - kz - \pi/4)$
f. $\mathcal{E}_x(z,t) = 3\cos(\omega t + kz - \pi/8)$, $\mathcal{E}_y(z,t) = 4\cos(\omega t + kz + \pi/8)$
g. $\mathcal{E}_x(z,t) = 4\cos(\omega t - kz + \pi/4)$, $\mathcal{E}_y(z,t) = 3\cos(\omega t - kz - \pi/2)$
h. $\mathcal{E}_x(z,t) = 3\cos(\omega t + kz - \pi/2)$, $\mathcal{E}_y(z,t) = 4\cos(\omega t + kz + \pi/4)$

Since these are either forward or backward waves, the corresponding magnetic fields are obtained by using the formula $\boldsymbol{\mathcal{H}}(z,t) = \pm\hat{\mathbf{z}}\times\boldsymbol{\mathcal{E}}(z,t)/\eta$. This gives the x, y components:

(cases a, c, e, g): $\mathcal{H}_x(z,t) = -\frac{1}{\eta}\mathcal{E}_y(z,t)$, $\mathcal{H}_y(z,t) = \frac{1}{\eta}\mathcal{E}_x(z,t)$

(cases b, d, f, h): $\mathcal{H}_x(z,t) = \frac{1}{\eta}\mathcal{E}_y(z,t)$, $\mathcal{H}_y(z,t) = -\frac{1}{\eta}\mathcal{E}_x(z,t)$

To determine the polarization vectors, we evaluate the electric fields at $z = 0$:

a. $\mathcal{E}_x(t) = 3\cos(\omega t - \pi/2)$, $\mathcal{E}_y(t) = 0$
b. $\mathcal{E}_x(t) = 3\cos(\omega t)$, $\mathcal{E}_y(t) = 4\cos(\omega t)$
c. $\mathcal{E}_x(t) = 4\cos(\omega t + \pi)$, $\mathcal{E}_y(t) = 3\cos(\omega t)$
d. $\mathcal{E}_x(t) = 3\cos(\omega t + \pi/3)$, $\mathcal{E}_y(t) = 3\cos(\omega t)$
e. $\mathcal{E}_x(t) = 4\cos(\omega t)$, $\mathcal{E}_y(t) = 3\cos(\omega t - \pi/4)$
f. $\mathcal{E}_x(t) = 3\cos(\omega t - \pi/8)$, $\mathcal{E}_y(t) = 4\cos(\omega t + \pi/8)$
g. $\mathcal{E}_x(t) = 4\cos(\omega t + \pi/4)$, $\mathcal{E}_y(t) = 3\cos(\omega t - \pi/2)$
h. $\mathcal{E}_x(t) = 3\cos(\omega t - \pi/2)$, $\mathcal{E}_y(t) = 4\cos(\omega t + \pi/4)$

The polarization ellipse parameters A, B, and $\phi = \phi_a - \phi_b$, as well as the computed semi-major axes A', B', tilt angle θ, sense of rotation of the electric field, and polarization type are given below:

case	A	B	ϕ	A'	B'	θ	rotation	polarization
a.	3	0	-90°	3	0	0°	→	linear/forward
b.	3	4	0°	0	5	-36.87°	↗	linear/backward
c.	4	3	180°	5	0	-36.87°	↖	linear/forward
d.	3	3	60°	3.674	2.121	45°	↺	left/backward
e.	4	3	45°	4.656	1.822	33.79°	↺	right/forward
f.	3	4	-45°	1.822	4.656	-33.79°	↻	right/backward
g.	4	3	135°	4.656	1.822	-33.79°	↺	right/forward
h.	3	4	-135°	1.822	4.656	33.79°	↻	right/backward

In the linear case (b), the polarization ellipse collapses along its A'-axis ($A' = 0$) and becomes a straight line along its B'-axis. The tilt angle θ still measures the angle of the A'-axis from the x-axis. The actual direction of the electric field will be $90^{\circ} - 36.87^{\circ} = 53.13^{\circ}$, which is equal to the slope angle, $\operatorname{atan}(B/A) = \operatorname{atan}(4/3) = 53.13^{\circ}$.

In case (c), the ellipse collapses along its B'-axis. Therefore, θ coincides with the angle of the slope of the electric field vector, that is, $\operatorname{atan}(-B/A) = \operatorname{atan}(-3/4) = -36.87^{\circ}$. □

With the understanding that θ always represents the slope of the A'-axis (whether collapsed or not, major or minor), Eqs. (2.5.5) and (2.5.6) correctly calculate all the special cases, except when $A = B$, which has tilt angle and semi-axes:

$$\theta = 45^{\circ}\,, \quad A' = A\sqrt{1 + \cos\phi}\,, \quad B' = A\sqrt{1 - \cos\phi} \tag{2.5.10}$$

The MATLAB function `ellipse.m` calculates the ellipse semi-axes and tilt angle, A', B', θ, given the parameters A, B, ϕ. It has usage:

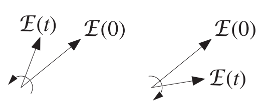
```
[a,b,th] = ellipse(A,B,phi)        % polarization ellipse parameters
```

For example, the function will return the values of the A', B', θ columns of the previous example, if it is called with the inputs:

```
A   = [3, 3, 4, 3, 4, 3, 4, 3]';
B   = [0, 4, 3, 3, 3, 4, 3, 4]';
phi = [-90, 0, 180, 60, 45, -45, 135, -135]';
```

To determine quickly the sense of rotation around the polarization ellipse, we use the rule that the rotation will be counterclockwise if the phase difference $\phi = \phi_a - \phi_b$ is such that $\sin\phi > 0$, and clockwise, if $\sin\phi < 0$. This can be seen by considering the electric field at time $t = 0$ and at a neighboring time t. Using Eq. (2.5.3), we have:

$$\begin{aligned} \boldsymbol{\mathcal{E}}(0) &= \hat{\mathbf{x}}\, A\cos\phi_a + \hat{\mathbf{y}}\, B\cos\phi_b \\ \boldsymbol{\mathcal{E}}(t) &= \hat{\mathbf{x}}\, A\cos(\omega t + \phi_a) + \hat{\mathbf{y}}\, B\cos(\omega t + \phi_b) \end{aligned}$$



The sense of rotation may be determined from the cross-product $\boldsymbol{\mathcal{E}}(0)\times\boldsymbol{\mathcal{E}}(t)$. If the rotation is counterclockwise, this vector will point towards the positive z-direction, and otherwise, it will point towards the negative z-direction. It follows easily that:

$$\boldsymbol{\mathcal{E}}(0)\times\boldsymbol{\mathcal{E}}(t) = \hat{\mathbf{z}}\, AB\sin\phi\sin\omega t \tag{2.5.11}$$

Thus, for t small and positive (such that $\sin\omega t > 0$), the direction of the vector $\boldsymbol{\mathcal{E}}(0)\times\boldsymbol{\mathcal{E}}(t)$ is determined by the sign of $\sin\phi$.

2.6 Uniform Plane Waves in Lossy Media

We saw in Sec. 1.14 that power losses may arise because of conduction and/or material polarization. A wave propagating in a lossy medium will set up a conduction current $\mathbf{J}_{\text{cond}} = \sigma\mathbf{E}$ and a displacement (polarization) current $\mathbf{J}_{\text{disp}} = j\omega\mathbf{D} = j\omega\epsilon_d\mathbf{E}$. Both currents will cause ohmic losses. The total current is the sum:

$$\mathbf{J}_{\text{tot}} = \mathbf{J}_{\text{cond}} + \mathbf{J}_{\text{disp}} = (\sigma + j\omega\epsilon_d)\mathbf{E} = j\omega\epsilon_c\mathbf{E}$$

where ϵ_c is the effective complex dielectric constant introduced in Eq. (1.14.2):

$$j\omega\epsilon_c = \sigma + j\omega\epsilon_d \quad\Rightarrow\quad \epsilon_c = \epsilon_d - j\,\frac{\sigma}{\omega} \tag{2.6.1}$$

The quantities σ, ϵ_d may be complex-valued and frequency-dependent. However, we will assume that over the desired frequency band of interest, the conductivity σ is real-valued; the permittivity of the dielectric may be assumed to be complex, $\epsilon_d = \epsilon_d' - j\epsilon_d''$. Thus, the effective ϵ_c has real and imaginary parts:

$$\boxed{\epsilon_c = \epsilon' - j\epsilon'' = \epsilon_d' - j\left(\epsilon_d'' + \frac{\sigma}{\omega}\right)} \tag{2.6.2}$$

Power losses arise from the non-zero imaginary part ϵ''. We recall from Eq. (1.14.5) that the time-averaged ohmic power losses per unit volume are given by:

$$\frac{dP_{\text{loss}}}{dV} = \frac{1}{2}\operatorname{Re}\big[\mathbf{J}_{\text{tot}}\cdot\mathbf{E}^*\big] = \frac{1}{2}\omega\epsilon''\,|\mathbf{E}|^2 = \frac{1}{2}(\sigma + \omega\epsilon_d'')\,|\mathbf{E}|^2 \tag{2.6.3}$$

Uniform plane waves propagating in such lossy medium will satisfy Maxwell's equations (1.9.2), with the right-hand side of Ampère's law given by $\mathbf{J}_{\text{tot}} = \mathbf{J} + j\omega\mathbf{D} = j\omega\epsilon_c\mathbf{E}$.

The assumption of uniformity ($\partial_x = \partial_y = 0$), will imply again that the fields $\mathbf{E}, \mathbf{H}$ are transverse to the direction $\hat{\mathbf{z}}$. Then, Faraday's and Ampère's equations become:

$$\begin{aligned} \nabla\times\mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla\times\mathbf{H} &= j\omega\epsilon_c\mathbf{E} \end{aligned} \quad\Rightarrow\quad \begin{aligned} \hat{\mathbf{z}}\times\partial_z\mathbf{E} &= -j\omega\mu\mathbf{H} \\ \hat{\mathbf{z}}\times\partial_z\mathbf{H} &= j\omega\epsilon_c\mathbf{E} \end{aligned} \tag{2.6.4}$$

These may be written in a more convenient form by introducing the complex wavenumber k_c and complex characteristic impedance η_c defined by:

$$\boxed{k_c = \omega\sqrt{\mu\epsilon_c}\,, \quad \eta_c = \sqrt{\frac{\mu}{\epsilon_c}}} \tag{2.6.5}$$

They correspond to the usual definitions $k = \omega/c = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\mu/\epsilon}$ with the replacement $\epsilon \to \epsilon_c$. Noting that $\omega\mu = k_c\eta_c$ and $\omega\epsilon_c = k_c/\eta_c$, Eqs. (2.6.4) may be written in the following form (using the orthogonality property $\hat{\mathbf{z}} \cdot \mathbf{E} = 0$ and the BAC-CAB rule on the first equation):

$$\frac{\partial}{\partial z}\begin{bmatrix} \mathbf{E} \\ \eta_c \mathbf{H}\times\hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 0 & -jk_c \\ -jk_c & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E} \\ \eta_c \mathbf{H}\times\hat{\mathbf{z}} \end{bmatrix} \tag{2.6.6}$$

To decouple them, we introduce the forward and backward electric fields:

$$\begin{aligned} \mathbf{E}_+ &= \frac{1}{2}(\mathbf{E} + \eta_c \mathbf{H}\times\hat{\mathbf{z}}) \\ \mathbf{E}_- &= \frac{1}{2}(\mathbf{E} - \eta_c \mathbf{H}\times\hat{\mathbf{z}}) \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \mathbf{E} &= \mathbf{E}_+ + \mathbf{E}_- \\ \mathbf{H} &= \frac{1}{\eta_c}\hat{\mathbf{z}}\times[\mathbf{E}_+ - \mathbf{E}_-] \end{aligned} \tag{2.6.7}$$

Then, Eqs. (2.6.6) may be replaced by the equivalent system:

$$\frac{\partial}{\partial z}\begin{bmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{bmatrix} = \begin{bmatrix} -jk_c & 0 \\ 0 & jk_c \end{bmatrix}\begin{bmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{bmatrix} \tag{2.6.8}$$

with solutions:

$$\mathbf{E}_\pm(z) = \mathbf{E}_{0\pm}e^{\mp jk_c z}, \qquad \text{where} \quad \hat{\mathbf{z}}\cdot\mathbf{E}_{0\pm} = 0 \tag{2.6.9}$$

Thus, the propagating electric and magnetic fields are linear combinations of forward and backward components:

$$\boxed{\begin{aligned} \mathbf{E}(z) &= \mathbf{E}_{0+}e^{-jk_c z} + \mathbf{E}_{0-}e^{jk_c z} \\ \mathbf{H}(z) &= \frac{1}{\eta_c}\hat{\mathbf{z}}\times[\mathbf{E}_{0+}e^{-jk_c z} - \mathbf{E}_{0-}e^{jk_c z}] \end{aligned}} \tag{2.6.10}$$

In particular, for a forward-moving wave we have:

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-jk_c z}, \quad \mathbf{H}(z) = \mathbf{H}_0 e^{-jk_c z}, \quad \text{with} \quad \hat{\mathbf{z}}\cdot\mathbf{E}_0 = 0, \quad \mathbf{H}_0 = \frac{1}{\eta_c}\hat{\mathbf{z}}\times\mathbf{E}_0 \tag{2.6.11}$$

Eqs. (2.6.10) are the same as in the lossless case but with the replacements $k \to k_c$ and $\eta \to \eta_c$. The lossless case is obtained in the limit of a purely real-valued ϵ_c.

Because k_c is complex-valued, we define the *phase* and *attenuation* constants β and α as the real and imaginary parts of k_c, that is,

$$k_c = \beta - j\alpha = \omega\sqrt{\mu(\epsilon' - j\epsilon'')} \tag{2.6.12}$$

We may also define a *complex refractive index* $n_c = k_c/k_0$ that measures k_c relative to its free-space value $k_0 = \omega/c_0 = \omega\sqrt{\mu_0\epsilon_0}$. For a non-magnetic medium, we have:

$$n_c = \frac{k_c}{k_0} = \sqrt{\frac{\epsilon_c}{\epsilon_0}} = \sqrt{\frac{\epsilon' - j\epsilon''}{\epsilon_0}} \equiv n_r - jn_i \tag{2.6.13}$$

where n_r, n_i are the real and imaginary parts of n_c. The quantity n_i is called the *extinction coefficient* and n_r, the refractive index. Another commonly used notation is the *propagation constant* γ defined by:

$$\gamma = jk_c = \alpha + j\beta \tag{2.6.14}$$

It follows from $\gamma = \alpha + j\beta = jk_c = jk_0 n_c = jk_0(n_r - jn_i)$ that $\beta = k_0 n_r$ and $\alpha = k_0 n_i$. The nomenclature about phase and attenuation constants has its origins in the propagation factor $e^{-jk_c z}$. We can write it in the alternative forms:

$$\boxed{e^{-jk_c z} = e^{-\gamma z} = e^{-\alpha z}e^{-j\beta z} = e^{-k_0 n_i z}e^{-jk_0 n_r z}} \tag{2.6.15}$$

Thus, the wave amplitudes *attenuate exponentially* with the factor $e^{-\alpha z}$, and oscillate with the phase factor $e^{-j\beta z}$. The energy of the wave attenuates by the factor $e^{-2\alpha z}$, as can be seen by computing the Poynting vector. Because $e^{-jk_c z}$ is no longer a pure phase factor and η_c is not real, we have for the forward-moving wave of Eq. (2.6.11):

$$\begin{aligned} \boldsymbol{\mathcal{P}}(z) &= \frac{1}{2}\text{Re}[\mathbf{E}(z)\times\mathbf{H}^*(z)] = \frac{1}{2}\text{Re}\left[\frac{1}{\eta_c^*}\mathbf{E}_0\times(\hat{\mathbf{z}}\times\mathbf{E}_0^*)e^{-(\alpha+j\beta)z}e^{-(\alpha-j\beta)z}\right] \\ &= \hat{\mathbf{z}}\,\frac{1}{2}\text{Re}(\eta_c^{-1})\,|\mathbf{E}_0|^2 e^{-2\alpha z} = \hat{\mathbf{z}}\,\mathcal{P}(0)e^{-2\alpha z} = \hat{\mathbf{z}}\,\mathcal{P}(z) \end{aligned}$$

Thus, the power per unit area flowing past the point z in the forward z-direction will be:

$$\boxed{\mathcal{P}(z) = \mathcal{P}(0)e^{-2\alpha z}} \tag{2.6.16}$$

The quantity $\mathcal{P}(0)$ is the power per unit area flowing past the point $z = 0$. Denoting the real and imaginary parts of η_c by η', η'', so that, $\eta_c = \eta' + j\eta''$, and noting that $|\mathbf{E}_0| = |\eta_c \mathbf{H}_0\times\hat{\mathbf{z}}| = |\eta_c||\mathbf{H}_0|$, we may express $\mathcal{P}(0)$ in the equivalent forms:

$$\mathcal{P}(0) = \frac{1}{2}\text{Re}(\eta_c^{-1})\,|\mathbf{E}_0|^2 = \frac{1}{2}\eta'\,|\mathbf{H}_0|^2 \tag{2.6.17}$$

The attenuation coefficient α is measured in *nepers per meter.* However, a more practical way of expressing the power attenuation is in dB per meter. Taking logs of Eq. (2.6.16), we have for the dB attenuation at z, relative to $z = 0$:

$$A_{\text{dB}}(z) = -10\log_{10}\left[\frac{\mathcal{P}(z)}{\mathcal{P}(0)}\right] = 20\log_{10}(e)\,\alpha z = 8.686\,\alpha z \tag{2.6.18}$$

where we used the numerical value $20\log_{10}e = 8.686$. Thus, the quantity $\alpha_{\text{dB}} = 8.686\,\alpha$ is the attenuation in *dB per meter*:

$$\alpha_{\text{dB}} = 8.686\,\alpha \qquad (\text{dB/m}) \tag{2.6.19}$$

Another way of expressing the power attenuation is by means of the so-called *penetration* or *skin depth* defined as the inverse of α:

$$\boxed{\delta = \frac{1}{\alpha}} \qquad (\text{skin depth}) \tag{2.6.20}$$

Then, Eq. (2.6.18) can be rewritten in the form:

$$A_{\text{dB}}(z) = 8.686\,\frac{z}{\delta} \qquad (\text{attenuation in dB}) \tag{2.6.21}$$

This gives rise to the so-called "9-dB per delta" rule, that is, every time z is increased by a distance δ, the attenuation increases by $8.686 \simeq 9$ dB.

A useful way to represent Eq. (2.6.16) in practice is to consider its infinitesimal version obtained by differentiating it with respect to z and solving for α:

$$\mathcal{P}'(z) = -2\alpha\mathcal{P}(0)e^{-2\alpha z} = -2\alpha\mathcal{P}(z) \quad\Rightarrow\quad \alpha = -\frac{\mathcal{P}'(z)}{2\mathcal{P}(z)}$$

The quantity $\mathcal{P}'_{\text{loss}} = -\mathcal{P}'$ represents the power lost from the wave per unit length (in the propagation direction.) Thus, the attenuation coefficient is the ratio of the power loss per unit length to twice the power transmitted:

$$\boxed{\alpha = \frac{P'_{\text{loss}}}{2P_{\text{transm}}}} \qquad \text{(attenuation coefficient)} \tag{2.6.22}$$

If there are several physical mechanisms for the power loss, then α becomes the *sum* over all possible cases. For example, in a waveguide or a coaxial cable filled with a slightly lossy dielectric, power will be lost because of the small conduction/polarization currents set up within the dielectric and also because of the ohmic losses in the walls of the guiding conductors, so that the total α will be $\alpha = \alpha_{\text{diel}} + \alpha_{\text{walls}}$.

Next, we verify that the exponential loss of power from the propagating wave is due to ohmic heat losses. In Fig. 2.6.1, we consider a volume $dV = l\,dA$ of area dA and length l along the z-direction.

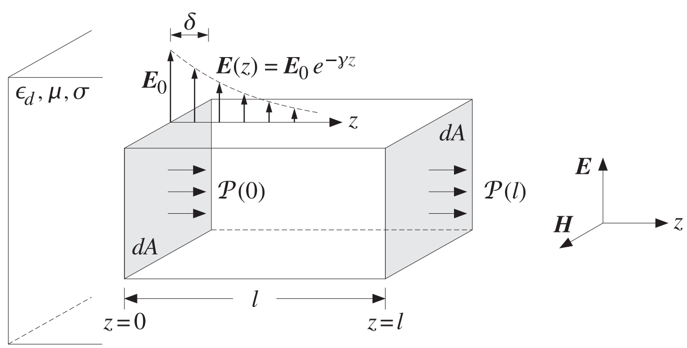


Fig. 2.6.1 Power flow in lossy dielectric.

From the definition of $\mathcal{P}(z)$ as power flow per unit area, it follows that the power entering the area dA at $z = 0$ will be $dP_{\text{in}} = \mathcal{P}(0)\,dA$, and the power leaving that area at $z = l$, $dP_{\text{out}} = \mathcal{P}(l)\,dA$. The difference $dP_{\text{loss}} = dP_{\text{in}} - dP_{\text{out}} = \big[\mathcal{P}(0) - \mathcal{P}(l)\big]dA$ will be the power lost from the wave within the volume $l\,dA$. Because $\mathcal{P}(l) = \mathcal{P}(0)e^{-2\alpha l}$, we have for the power loss per unit area:

$$\frac{dP_{\text{loss}}}{dA} = \mathcal{P}(0) - \mathcal{P}(l) = \mathcal{P}(0)\big(1 - e^{-2\alpha l}\big) = \frac{1}{2}\text{Re}\big(\eta_c^{-1}\big)\,|\boldsymbol{E}_0|^2\big(1 - e^{-2\alpha l}\big) \tag{2.6.23}$$

On the other hand, according to Eq. (2.6.3), the ohmic power loss per unit volume will be $\omega\epsilon''|\boldsymbol{E}(z)|^2/2$. Integrating this quantity from $z = 0$ to $z = l$ will give the total ohmic losses within the volume $l\,dA$ of Fig. 2.6.1. Thus, we have:

$$dP_{\text{ohmic}} = \frac{1}{2}\omega\epsilon''\int_0^l |\boldsymbol{E}(z)|^2\,dz\,dA = \frac{1}{2}\omega\epsilon''\left[\int_0^l |\boldsymbol{E}_0|^2 e^{-2\alpha z}\,dz\right]dA\,, \qquad \text{or,}$$

$$\frac{dP_{\text{ohmic}}}{dA} = \frac{\omega\epsilon''}{4\alpha}\,|\boldsymbol{E}_0|^2\big(1 - e^{-2\alpha l}\big) \tag{2.6.24}$$

Are the two expressions in Eqs. (2.6.23) and (2.6.24) equal? The answer is yes, as follows from the following relationship among the quantities $\eta_c, \epsilon'', \alpha$ (see Problem 2.17):

$$\boxed{\text{Re}\big(\eta_c^{-1}\big) = \frac{\omega\epsilon''}{2\alpha}} \tag{2.6.25}$$

Thus, the power lost from the wave is entirely accounted for by the ohmic losses within the propagation medium. The equality of (2.6.23) and (2.6.24) is an example of the more general relationship proved in Problem 1.5.

In the limit $l \to \infty$, we have $\mathcal{P}(l) \to 0$, so that $dP_{\text{ohmic}}/dA = \mathcal{P}(0)$, which states that all the power that enters at $z = 0$ will be dissipated into heat inside the semi-infinite medium. Using Eq. (2.6.17), we summarize this case:

$$\boxed{\frac{dP_{\text{ohmic}}}{dA} = \frac{1}{2}\text{Re}\big(\eta_c^{-1}\big)\,|\boldsymbol{E}_0|^2 = \frac{1}{2}\eta'\,|\boldsymbol{H}_0|^2} \qquad \text{(ohmic losses)} \tag{2.6.26}$$

This result will be used later on to calculate ohmic losses of waves incident on lossy dielectric or conductor surfaces, as well as conductor losses in waveguide and transmission line problems.

Example 2.6.1: The absorption coefficient α of water reaches a minimum over the visible spectrum—a fact undoubtedly responsible for why the visible spectrum is visible.

Recent measurements [136] of the absorption coefficient show that it starts at about 0.01 nepers/m at 380 nm (violet), decreases to a minimum value of 0.0044 nepers/m at 418 nm (blue), and then increases steadily reaching the value of 0.5 nepers/m at 600 nm (red). Determine the penetration depth δ in meters, for each of the three wavelengths.

Determine the depth in meters at which the light intensity has decreased to 1/10th its value at the surface of the water. Repeat, if the intensity is decreased to 1/100th its value.

Solution: The penetration depths $\delta = 1/\alpha$ are:

$$\delta = 100,\ 227.3,\ 2\text{ m} \quad\text{for}\quad \alpha = 0.01,\ 0.0044,\ 0.5\text{ nepers/m}$$

Using Eq. (2.6.21), we may solve for the depth $z = (A/8.9696)\delta$. Since a decrease of the light intensity (power) by a factor of 10 is equivalent to $A = 10$ dB, we find $z = (10/8.9696)\delta = 1.128\,\delta$, which gives: $z = 112.8,\ 256.3,\ 2.3$ m. A decrease by a factor of $100 = 10^{20/10}$ corresponds to $A = 20$ dB, effectively doubling the above depths. □

Example 2.6.2: A microwave oven operating at 2.45 GHz is used to defrost a frozen food having complex permittivity $\epsilon_c = (4-j)\epsilon_0$ farad/m. Determine the strength of the electric field at a depth of 1 cm and express it in dB and as a percentage of its value at the surface. Repeat if $\epsilon_c = (45-15j)\epsilon_0$ farad/m.

Solution: The free-space wavenumber is $k_0 = \omega\sqrt{\mu_0\epsilon_0} = 2\pi f/c_0 = 2\pi(2.45\times 10^9)/(3\times 10^8) = 51.31$ rad/m. Using $k_c = \omega\sqrt{\mu_0\epsilon_c} = k_0\sqrt{\epsilon_c/\epsilon_0}$, we calculate the wavenumbers:

$$k_c = \beta - j\alpha = 51.31\sqrt{4-j} = 51.31(2.02-0.25j) = 103.41 - 12.73j\ \mathrm{m}^{-1}$$
$$k_c = \beta - j\alpha = 51.31\sqrt{45-15j} = 51.31(6.80-1.10j) = 348.84 - 56.61j\ \mathrm{m}^{-1}$$

The corresponding attenuation constants and penetration depths are:

$$\alpha = 12.73 \text{ nepers/m}, \qquad \delta = 7.86 \text{ cm}$$
$$\alpha = 56.61 \text{ nepers/m}, \qquad \delta = 1.77 \text{ cm}$$

It follows that the attenuations at 1 cm will be in dB and in absolute units:

$$A = 8.686\,z/\delta = 1.1 \text{ dB}, \qquad 10^{-A/20} = 0.88$$
$$A = 8.686\,z/\delta = 4.9 \text{ dB}, \qquad 10^{-A/20} = 0.57$$

Thus, the fields at a depth of 1 cm are 88% and 57% of their values at the surface. The complex permittivities of some foods may be found in [137]. □

A convenient way to characterize the degree of ohmic losses is by means of the *loss tangent*, originally defined in Eq. (1.14.8). Here, we set:

$$\tau = \tan\theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma + \omega\epsilon_d''}{\omega\epsilon_d'} \tag{2.6.27}$$

Then, $\epsilon_c = \epsilon' - j\epsilon'' = \epsilon'(1-j\tau) = \epsilon_d'(1-j\tau)$. Therefore, k_c, η_c may be written as:

$$k_c = \omega\sqrt{\mu\epsilon_d'}\,(1-j\tau)^{1/2}, \quad \eta_c = \sqrt{\frac{\mu}{\epsilon_d'}}\,(1-j\tau)^{-1/2} \tag{2.6.28}$$

The quantities $c_d = 1/\sqrt{\mu\epsilon_d'}$ and $\eta_d = \sqrt{\mu/\epsilon_d'}$ would be the speed of light and characteristic impedance of an equivalent lossless dielectric with permittivity ϵ_d'.

In terms of the loss tangent, we may characterize *weakly* lossy media versus *strongly* lossy ones by the conditions $\tau \ll 1$ versus $\tau \gg 1$, respectively. These conditions depend on the operating frequency ω:

$$\frac{\sigma + \omega\epsilon_d''}{\omega\epsilon_d'} \ll 1 \quad \text{versus} \quad \frac{\sigma + \omega\epsilon_d''}{\omega\epsilon_d'} \gg 1$$

The expressions (2.6.28) may be simplified considerably in these two limits. Using the small-x Taylor series expansion $(1+x)^{1/2} \simeq 1 + x/2$, we find in the weakly lossy case $(1-j\tau)^{1/2} \simeq 1 - j\tau/2$, and similarly, $(1-j\tau)^{-1/2} \simeq 1 + j\tau/2$.

On the other hand, if $\tau \gg 1$, we may approximate $(1-j\tau)^{1/2} \simeq (-j\tau)^{1/2} = e^{-j\pi/4}\tau^{1/2}$, where we wrote $(-j)^{1/2} = (e^{-j\pi/2})^{1/2} = e^{-j\pi/4}$. Similarly, $(1-j\tau)^{-1/2} \simeq e^{j\pi/4}\tau^{-1/2}$. Thus, we summarize the two limits:

$$(1-j\tau)^{1/2} = \begin{cases} 1 - j\dfrac{\tau}{2}, & \text{if } \tau \ll 1 \\ e^{-j\pi/4}\tau^{1/2} = (1-j)\sqrt{\dfrac{\tau}{2}}, & \text{if } \tau \gg 1 \end{cases} \tag{2.6.29}$$

$$(1-j\tau)^{-1/2} = \begin{cases} 1 + j\dfrac{\tau}{2}, & \text{if } \tau \ll 1 \\ e^{j\pi/4}\tau^{-1/2} = (1+j)\sqrt{\dfrac{1}{2\tau}}, & \text{if } \tau \gg 1 \end{cases} \tag{2.6.30}$$

2.7 Propagation in Weakly Lossy Dielectrics

In the weakly lossy case, the propagation parameters k_c, η_c become:

$$\begin{aligned} k_c &= \beta - j\alpha = \omega\sqrt{\mu\epsilon_d'}\left(1 - j\frac{\tau}{2}\right) = \omega\sqrt{\mu\epsilon_d'}\left(1 - j\frac{\sigma + \omega\epsilon_d''}{2\omega\epsilon_d'}\right) \\ \eta_c &= \eta' + j\eta'' = \sqrt{\frac{\mu}{\epsilon_d'}}\left(1 + j\frac{\tau}{2}\right) = \sqrt{\frac{\mu}{\epsilon_d'}}\left(1 + j\frac{\sigma + \omega\epsilon_d''}{2\omega\epsilon_d'}\right) \end{aligned} \tag{2.7.1}$$

Thus, the phase and attenuation constants are:

$$\beta = \omega\sqrt{\mu\epsilon_d'} = \frac{\omega}{c_d}, \quad \alpha = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon_d'}}(\sigma + \omega\epsilon_d'') = \frac{1}{2}\eta_d(\sigma + \omega\epsilon_d'') \tag{2.7.2}$$

For a slightly conducting dielectric with $\epsilon_d'' = 0$ and a small conductivity σ, Eq. (2.7.2) implies that the attenuation coefficient α is frequency-independent in this limit.

Example 2.7.1: Seawater has $\sigma = 4$ Siemens/m and $\epsilon_d = 81\epsilon_0$ (so that $\epsilon_d' = 81\epsilon_0$, $\epsilon_d'' = 0$.) Then, $n_d = \sqrt{\epsilon_d/\epsilon_0} = 9$, and $c_d = c_0/n_d = 0.33\times 10^8$ m/sec and $\eta_d = \eta_0/n_d = 377/9 = 41.89\ \Omega$. The attenuation coefficient (2.7.2) will be:

$$\alpha = \frac{1}{2}\eta_d\sigma = \frac{1}{2}41.89\times 4 = 83.78 \text{ nepers/m} \quad \Rightarrow \quad \alpha_{\mathrm{dB}} = 8.686\,\alpha = 728 \text{ dB/m}$$

The corresponding skin depth is $\delta = 1/\alpha = 1.19$ cm. This result assumes that $\sigma \ll \omega\epsilon_d$, which can be written in the form $\omega \gg \sigma/\epsilon_d$, or $f \gg f_0$, where $f_0 = \sigma/(2\pi\epsilon_d)$. Here, we have $f_0 = 888$ MHz. For frequencies $f \lesssim f_0$, we must use the exact equations (2.6.28). For example, we find:

$$f = 1 \text{ kHz}, \quad \alpha_{\mathrm{dB}} = 1.09 \text{ dB/m}, \quad \delta = 7.96 \text{ m}$$
$$f = 1 \text{ MHz}, \quad \alpha_{\mathrm{dB}} = 34.49 \text{ dB/m}, \quad \delta = 25.18 \text{ cm}$$
$$f = 1 \text{ GHz}, \quad \alpha_{\mathrm{dB}} = 672.69 \text{ dB/m}, \quad \delta = 1.29 \text{ cm}$$

Such extremely large attenuations explain why communication with submarines is impossible at high RF frequencies. □

2.8 Propagation in Good Conductors

A conductor is characterized by a large value of its conductivity σ, while its dielectric constant may be assumed to be real-valued $\epsilon_d = \epsilon$ (typically equal to ϵ_0.) Thus, its complex permittivity and loss tangent will be:

$$\epsilon_c = \epsilon - j\frac{\sigma}{\omega} = \epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right), \quad \tau = \frac{\sigma}{\omega\epsilon} \tag{2.8.1}$$

A good conductor corresponds to the limit $\tau \gg 1$, or, $\sigma \gg \omega\epsilon$. Using the approximations of Eqs. (2.6.29) and (2.6.30), we find for the propagation parameters k_c, η_c:

$$\begin{aligned} k_c &= \beta - j\alpha = \omega\sqrt{\mu\epsilon}\sqrt{\frac{\tau}{2}}\,(1-j) = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1-j) \\ \eta_c &= \eta' + j\eta'' = \sqrt{\frac{\mu}{\epsilon}}\sqrt{\frac{1}{2\tau}}\,(1+j) = \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) \end{aligned} \tag{2.8.2}$$

Thus, the parameters β, α, δ are:

$$\boxed{\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}}\,, \quad \boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f\mu\sigma}}} \tag{2.8.3}$$

where we replaced $\omega = 2\pi f$. The complex characteristic impedance η_c can be written in the form $\eta_c = R_s(1+j)$, where R_s is called the *surface resistance* and is given by the equivalent forms (where $\eta = \sqrt{\mu/\epsilon}$):

$$\boxed{R_s = \eta\sqrt{\frac{\omega\epsilon}{2\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}} = \frac{\alpha}{\sigma} = \frac{1}{\sigma\delta}} \tag{2.8.4}$$

Example 2.8.1: For copper we have $\sigma = 5.8\times 10^7$ Siemens/m. The skin depth at frequency f is:

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = \frac{1}{\sqrt{\pi\cdot 4\pi\cdot 10^{-7}\cdot 5.8\cdot 10^7}}\,f^{-1/2} = 0.0661\,f^{-1/2} \qquad (f \text{ in Hz})$$

We find at frequencies of 1 kHz, 1 MHz, and 1 GHz:

$$\begin{aligned} f &= 1\text{ kHz}, & \delta &= 2.09\text{ mm} \\ f &= 1\text{ MHz}, & \delta &= 0.07\text{ mm} \\ f &= 1\text{ GHz}, & \delta &= 2.09\ \mu\text{m} \end{aligned}$$

Thus, the skin depth is extremely small for good conductors at RF. □

Because δ is so small, the fields will attenuate rapidly within the conductor, depending on distance like $e^{-\gamma z} = e^{-\alpha z}e^{-j\beta z} = e^{-z/\delta}e^{-j\beta z}$. The factor $e^{-z/\delta}$ effectively confines the fields to within a distance δ from the surface of the conductor.

This allows us to define equivalent "surface" quantities, such as surface current and surface impedance. With reference to Fig. 2.6.1, we define the surface current density by integrating the density $\boldsymbol{J}(z) = \sigma\boldsymbol{E}(z) = \sigma\boldsymbol{E}_0 e^{-\gamma z}$ over the top-side of the volume $l\,dA$, and taking the limit $l \to \infty$:

$$\boldsymbol{J}_s = \int_0^\infty \boldsymbol{J}(z)\,dz = \int_0^\infty \sigma\boldsymbol{E}_0 e^{-\gamma z}\,dz = \frac{\sigma}{\gamma}\boldsymbol{E}_0\,, \quad \text{or,}$$

$$\boldsymbol{J}_s = \frac{1}{Z_s}\,\boldsymbol{E}_0 \tag{2.8.5}$$

where we defined the *surface impedance* $Z_s = \gamma/\sigma$. In the good-conductor limit, Z_s is equal to η_c. Indeed, it follows from Eqs. (2.8.3) and (2.8.4) that:

$$Z_s = \frac{\gamma}{\sigma} = \frac{\alpha + j\beta}{\sigma} = \frac{\alpha}{\sigma}(1+j) = R_s(1+j) = \eta_c$$

Because $\boldsymbol{H}_0\times\hat{\mathbf{z}} = \boldsymbol{E}_0/\eta_c$, it follows that the surface current will be related to the magnetic field intensity at the surface of the conductor by:

$$\boldsymbol{J}_s = \boldsymbol{H}_0\times\hat{\mathbf{z}} = \hat{\mathbf{n}}\times\boldsymbol{H}_0 \tag{2.8.6}$$

where $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ is the outward normal to the conductor. The meaning of $\boldsymbol{J}_s$ is that it represents the current flowing in the direction of $\boldsymbol{E}_0$ per unit length measured along the perpendicular direction to $\boldsymbol{E}_0$, that is, the $\boldsymbol{H}_0$-direction. It has units of A/m.

The total amount of ohmic losses per unit surface area of the conductor may be calculated from Eq. (2.6.26), which reads in this case:

$$\boxed{\frac{dP_{\text{ohmic}}}{dA} = \frac{1}{2}R_s|\boldsymbol{H}_0|^2 = \frac{1}{2}R_s|\boldsymbol{J}_s|^2} \quad \text{(ohmic loss per unit conductor area)} \tag{2.8.7}$$

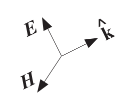
2.9 Propagation in Oblique Directions

So far we considered waves propagating towards the z-direction. For single-frequency uniform plane waves propagating in some arbitrary direction in a lossless medium, the propagation factor is obtained by the substitution:

$$e^{-jkz} \to e^{-j\,\boldsymbol{k}\cdot\mathbf{r}}$$

where $\boldsymbol{k} = k\hat{\mathbf{k}}$, with $k = \omega\sqrt{\mu\epsilon} = \omega/c$ and $\hat{\mathbf{k}}$ is a *unit vector* in the direction of propagation. The fields take the form:

$$\begin{aligned} \boldsymbol{E}(\mathbf{r},t) &= \boldsymbol{E}_0 e^{j\omega t - j\,\boldsymbol{k}\cdot\mathbf{r}} \\ \boldsymbol{H}(\mathbf{r},t) &= \boldsymbol{H}_0 e^{j\omega t - j\,\boldsymbol{k}\cdot\mathbf{r}} \end{aligned} \tag{2.9.1}$$



where $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ are constant vectors transverse to $\hat{\mathbf{k}}$, that is, $\hat{\mathbf{k}}\cdot\boldsymbol{E}_0 = \hat{\mathbf{k}}\cdot\boldsymbol{H}_0 = 0$, such that:

$$\boldsymbol{H}_0 = \frac{1}{\omega\mu}\,\boldsymbol{k}\times\boldsymbol{E}_0 = \frac{1}{\eta}\,\hat{\mathbf{k}}\times\boldsymbol{E}_0 \tag{2.9.2}$$

where $\eta = \sqrt{\mu/\epsilon}$. Thus, $\{\boldsymbol{E}, \boldsymbol{H}, \hat{\mathbf{k}}\}$ form a right-handed orthogonal system.

The solutions (2.9.1) can be derived from Maxwell's equations in a straightforward fashion. When the gradient operator acts on the above fields, it can be simplified into $\boldsymbol{\nabla} \to -j\boldsymbol{k}$. This follows from:

$$\nabla\left(e^{-j\boldsymbol{k}\cdot\mathbf{r}}\right) = -j\boldsymbol{k}\left(e^{-j\boldsymbol{k}\cdot\mathbf{r}}\right)$$

After canceling the common factor $e^{j\omega t - j\boldsymbol{k}\cdot\mathbf{r}}$, Maxwell's equations (2.1.1) take the form:

$$\begin{aligned}
-j\boldsymbol{k}\times\boldsymbol{E}_0 &= -j\omega\mu\boldsymbol{H}_0 & \qquad & \boldsymbol{k}\times\boldsymbol{E}_0 = \omega\mu\boldsymbol{H}_0\\
-j\boldsymbol{k}\times\boldsymbol{H}_0 &= j\omega\epsilon\boldsymbol{E}_0 & \Rightarrow \qquad & \boldsymbol{k}\times\boldsymbol{H}_0 = -\omega\epsilon\boldsymbol{E}_0\\
\boldsymbol{k}\cdot\boldsymbol{E}_0 &= 0 & & \boldsymbol{k}\cdot\boldsymbol{E}_0 = 0\\
\boldsymbol{k}\cdot\boldsymbol{H}_0 &= 0 & & \boldsymbol{k}\cdot\boldsymbol{H}_0 = 0
\end{aligned} \tag{2.9.3}$$

The last two imply that $\boldsymbol{E}_0, \boldsymbol{H}_0$ are transverse to $\boldsymbol{k}$. The other two can be decoupled by taking the cross product of the first equation with $\boldsymbol{k}$ and using the second equation:

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \omega\mu\,\boldsymbol{k}\times\boldsymbol{H}_0 = -\omega^2\mu\epsilon\,\boldsymbol{E}_0 \tag{2.9.4}$$

The left-hand side can be simplified using the BAC-CAB rule and $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$, that is, $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}_0) - \boldsymbol{E}_0(\boldsymbol{k}\cdot\boldsymbol{k}) = -(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{E}_0$. Thus, Eq. (2.9.4) becomes:

$$-(\boldsymbol{k}\cdot\boldsymbol{k})\,\boldsymbol{E}_0 = -\omega^2\mu\epsilon\,\boldsymbol{E}_0$$

Thus, we obtain the consistency condition:

$$\boxed{\boldsymbol{k}\cdot\boldsymbol{k} = \omega^2\mu\epsilon} \tag{2.9.5}$$

Defining $k = \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}} = |\boldsymbol{k}|$, we have $k = \omega\sqrt{\mu\epsilon}$. Using the relationship $\omega\mu = k\eta$ and defining the unit vector $\hat{\mathbf{k}} = \boldsymbol{k}/|\boldsymbol{k}| = \boldsymbol{k}/k$, the magnetic field is obtained from:

$$\boldsymbol{H}_0 = \frac{\boldsymbol{k}\times\boldsymbol{E}_0}{\omega\mu} = \frac{\boldsymbol{k}\times\boldsymbol{E}_0}{k\eta} = \frac{1}{\eta}\hat{\mathbf{k}}\times\boldsymbol{E}_0$$

The constant-phase (and constant-amplitude) wavefronts are the planes $\boldsymbol{k}\cdot\mathbf{r} =$ constant, or, $\hat{\mathbf{k}}\cdot\mathbf{r} =$ constant. They are the planes perpendicular to the propagation direction $\hat{\mathbf{k}}$.

As an example, consider a rotated coordinate system $\{x', y', z'\}$ in which the $z'x'$ axes are rotated by angle θ relative to the original zx axes, as shown in Fig. 2.9.1. Thus, the new coordinates and corresponding unit vectors will be:

$$\begin{aligned}
z' &= z\cos\theta + x\sin\theta, & \qquad \hat{\mathbf{z}}' &= \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{x}}\sin\theta\\
x' &= x\cos\theta - z\sin\theta, & \hat{\mathbf{x}}' &= \hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta\\
y' &= y, & \hat{\mathbf{y}}' &= \hat{\mathbf{y}}
\end{aligned} \tag{2.9.6}$$

We choose the propagation direction to be the new z-axis, that is, $\hat{\mathbf{k}} = \hat{\mathbf{z}}'$, so that the wave vector $\boldsymbol{k} = k\hat{\mathbf{k}} = k\hat{\mathbf{z}}'$ will have components $k_z = k\cos\theta$ and $k_x = k\sin\theta$:

$$\boldsymbol{k} = k\hat{\mathbf{k}} = k(\hat{\mathbf{z}}\cos\theta + \hat{\mathbf{x}}\sin\theta) = \hat{\mathbf{z}}k_z + \hat{\mathbf{x}}k_x$$

The propagation phase factor becomes:

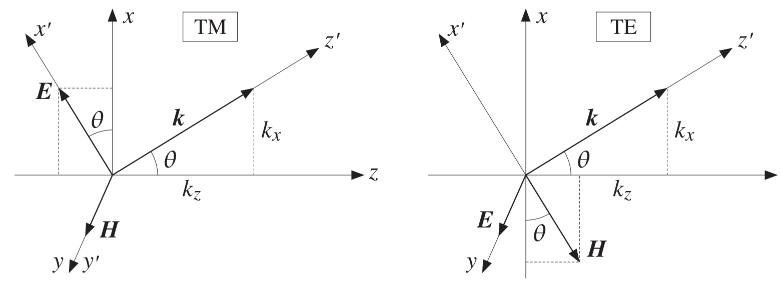


Fig. 2.9.1 TM and TE waves.

$$e^{-j\boldsymbol{k}\cdot\mathbf{r}} = e^{-j(k_z z + k_x x)} = e^{-jk(z\cos\theta + x\sin\theta)} = e^{-jkz'}$$

Because $\{\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{k}\}$ form a right-handed vector system, the electric field may have components along the new transverse (with respect to z') axes, that is, along x' and y. Thus, we may resolve $\boldsymbol{E}_0$ into the orthogonal directions:

$$\boldsymbol{E}_0 = \hat{\mathbf{x}}'A + \hat{\mathbf{y}}B = (\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)A + \hat{\mathbf{y}}B \tag{2.9.7}$$

The corresponding magnetic field will be $\boldsymbol{H}_0 = \hat{\mathbf{k}}\times\boldsymbol{E}_0/\eta = \hat{\mathbf{z}}'\times(\hat{\mathbf{x}}'A + \hat{\mathbf{y}}B)/\eta$. Using the relationships $\hat{\mathbf{z}}'\times\hat{\mathbf{x}}' = \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}'\times\hat{\mathbf{y}} = -\hat{\mathbf{x}}'$, we find:

$$\boldsymbol{H}_0 = \frac{1}{\eta}\left[\hat{\mathbf{y}}A - \hat{\mathbf{x}}'B\right] = \frac{1}{\eta}\left[\hat{\mathbf{y}}A - (\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)B\right] \tag{2.9.8}$$

The complete expressions for the fields are then:

$$\begin{aligned}
\boldsymbol{E}(\mathbf{r}, t) &= \left[(\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)A + \hat{\mathbf{y}}B\right]e^{j\omega t - jk(z\cos\theta + x\sin\theta)}\\
\boldsymbol{H}(\mathbf{r}, t) &= \frac{1}{\eta}\left[\hat{\mathbf{y}}A - (\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)B\right]e^{j\omega t - jk(z\cos\theta + x\sin\theta)}
\end{aligned} \tag{2.9.9}$$

Written with respect to the rotated coordinate system $\{x', y', z'\}$, the solutions become identical to those of Sec. 2.2:

$$\begin{aligned}
\boldsymbol{E}(\mathbf{r}, t) &= \left[\hat{\mathbf{x}}'A + \hat{\mathbf{y}}'B\right]e^{j\omega t - jkz'}\\
\boldsymbol{H}(\mathbf{r}, t) &= \frac{1}{\eta}\left[\hat{\mathbf{y}}'A - \hat{\mathbf{x}}'B\right]e^{j\omega t - jkz'}
\end{aligned} \tag{2.9.10}$$

They are uniform in the sense that they do not depend on the new transverse coordinates x', y'. The constant-phase planes are $z' = \hat{\mathbf{z}}'\cdot\mathbf{r} = z\cos\theta + x\sin\theta =$ const.

The polarization properties of the wave depend on the relative phases and amplitudes of the complex constants A, B, with the polarization ellipse lying on the $x'y'$ plane.

The A- and B-components of $\boldsymbol{E}_0$ are referred to as *transverse magnetic* (TM) and *transverse electric* (TE), respectively, where "transverse" is meant here with respect to the z-axis. The TE case has an *electric* field transverse to z; the TM case has a *magnetic* field transverse to z. Fig. 2.9.1 depicts these two cases separately.

This nomenclature arises in the context of plane waves incident obliquely on interfaces, where the xz plane is the plane of incidence and the interface is the xy plane. The TE and TM cases are also referred to as having "perpendicular" and "parallel" polarization vectors with respect to the plane of incidence, that is, the E-field is perpendicular or parallel to the xz plane.

We may define the concept of *transverse impedance* as the ratio of the transverse (with respect to z) components of the electric and magnetic fields. In particular, by analogy with the definitions of Sec. 2.4, we have:

$$\begin{aligned}\eta_{TM} &= \frac{E_x}{H_y} = \frac{A\cos\theta}{\dfrac{1}{\eta}A} = \eta\cos\theta\\ \eta_{TE} &= -\frac{E_y}{H_x} = \frac{B}{\dfrac{1}{\eta}B\cos\theta} = \frac{\eta}{\cos\theta}\end{aligned} \tag{2.9.11}$$

Such transverse impedances play an important role in describing the transfer matrices of dielectric slabs at oblique incidence. We discuss them further in Chap. 7.

2.10 Complex or Inhomogeneous Waves

The steps leading to the wave solution (2.9.1) do not preclude a *complex-valued* wavevector $\boldsymbol{k}$. For example, if the medium is lossy, we must replace $\{\eta, k\}$ by $\{\eta_c, k_c\}$, where $k_c = \beta - j\alpha$, resulting from a complex effective permittivity ϵ_c. If the propagation direction is defined by the unit vector $\hat{\mathbf{k}}$, chosen to be a *rotated* version of $\hat{\mathbf{z}}$, then the wavevector will be defined by $\boldsymbol{k} = k_c\hat{\mathbf{k}} = (\beta - j\alpha)\hat{\mathbf{k}}$. Because $k_c = \omega\sqrt{\mu\epsilon_c}$ and $\hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$, the vector $\boldsymbol{k}$ satisfies the consistency condition (2.9.5):

$$\boldsymbol{k}\cdot\boldsymbol{k} = k_c^2 = \omega^2\mu\epsilon_c \tag{2.10.1}$$

The propagation factor will be:

$$e^{-j\boldsymbol{k}\cdot\mathbf{r}} = e^{-jk_c\hat{\mathbf{k}}\cdot\mathbf{r}} = e^{-(\alpha+j\beta)\hat{\mathbf{k}}\cdot\mathbf{r}} = e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}}e^{-j\beta\hat{\mathbf{k}}\cdot\mathbf{r}}$$

The wave is still a uniform plane wave in the sense that the constant-amplitude planes, $\alpha\hat{\mathbf{k}}\cdot\mathbf{r} = \text{const.}$, and the constant-phase planes, $\beta\hat{\mathbf{k}}\cdot\mathbf{r} = \text{const.}$, coincide with each other—being the planes perpendicular to the propagation direction. For example, the rotated solution (2.9.10) becomes in the lossy case:

$$\begin{aligned}\boldsymbol{E}(\mathbf{r},t) &= [\hat{\mathbf{x}}'A + \hat{\mathbf{y}}'B]e^{j\omega t - jk_c z'} = [\hat{\mathbf{x}}'A + \hat{\mathbf{y}}'B]e^{j\omega t-(\alpha+j\beta)z'}\\ \boldsymbol{H}(\mathbf{r},t) &= \frac{1}{\eta_c}[\hat{\mathbf{y}}'A - \hat{\mathbf{x}}'B]e^{j\omega t - jk_c z'} = \frac{1}{\eta_c}[\hat{\mathbf{y}}'A - \hat{\mathbf{x}}'B]e^{j\omega t-(\alpha+j\beta)z'}\end{aligned} \tag{2.10.2}$$

In this solution, the real and imaginary parts of the wavevector $\boldsymbol{k} = \boldsymbol{\beta} - j\boldsymbol{\alpha}$ are collinear, that is, $\boldsymbol{\beta} = \beta\hat{\mathbf{k}}$ and $\boldsymbol{\alpha} = \alpha\hat{\mathbf{k}}$.

More generally, there exist solutions having a complex wavevector $\boldsymbol{k} = \boldsymbol{\beta} - j\boldsymbol{\alpha}$ such that $\boldsymbol{\beta}, \boldsymbol{\alpha}$ are not collinear. The propagation factor becomes now:

$$e^{-j\boldsymbol{k}\cdot\mathbf{r}} = e^{-(\boldsymbol{\alpha}+j\boldsymbol{\beta})\cdot\mathbf{r}} = e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}e^{-j\boldsymbol{\beta}\cdot\mathbf{r}} \tag{2.10.3}$$

If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are not collinear, such a wave will not be a uniform plane wave because the constant-amplitude planes, $\boldsymbol{\alpha}\cdot\mathbf{r} = \text{const.}$, and the constant-phase planes, $\boldsymbol{\beta}\cdot\mathbf{r} = \text{const.}$, will be different. The consistency condition $\boldsymbol{k}\cdot\boldsymbol{k} = k_c^2 = (\beta - j\alpha)^2$ splits into the following two conditions obtained by equating real and imaginary parts:

$$(\boldsymbol{\beta} - j\boldsymbol{\alpha})\cdot(\boldsymbol{\beta} - j\boldsymbol{\alpha}) = (\beta - j\alpha)^2 \quad\Leftrightarrow\quad \boxed{\begin{aligned}\boldsymbol{\beta}\cdot\boldsymbol{\beta} - \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} &= \beta^2 - \alpha^2\\ \boldsymbol{\beta}\cdot\boldsymbol{\alpha} &= \alpha\beta\end{aligned}} \tag{2.10.4}$$

With $\boldsymbol{E}_0$ chosen to satisfy $\boldsymbol{k}\cdot\boldsymbol{E}_0 = (\boldsymbol{\beta} - j\boldsymbol{\alpha})\cdot\boldsymbol{E}_0 = 0$, the magnetic field is computed from Eq. (2.9.2), $\boldsymbol{H}_0 = \boldsymbol{k}\times\boldsymbol{E}_0/\omega\mu = (\boldsymbol{\beta} - j\boldsymbol{\alpha})\times\boldsymbol{E}_0/\omega\mu$.

Let us look at an explicit construction. We choose $\boldsymbol{\beta}, \boldsymbol{\alpha}$ to lie on the xz plane of Fig. 2.9.1, and resolve them as $\boldsymbol{\beta} = \hat{\mathbf{z}}\beta_z + \hat{\mathbf{x}}\beta_x$ and $\boldsymbol{\alpha} = \hat{\mathbf{z}}\alpha_z + \hat{\mathbf{x}}\alpha_x$. Thus,

$$\boldsymbol{k} = \boldsymbol{\beta} - j\boldsymbol{\alpha} = \hat{\mathbf{z}}(\beta_z - j\alpha_z) + \hat{\mathbf{x}}(\beta_x - j\alpha_x) = \hat{\mathbf{z}}k_z + \hat{\mathbf{x}}k_x$$

Then, the propagation factor (2.10.3) and conditions (2.10.4) read explicitly:

$$\begin{aligned}&e^{-j\boldsymbol{k}\cdot\mathbf{r}} = e^{-(\alpha_z z+\alpha_x x)}e^{-j(\beta_z z+\beta_x x)}\\ &\beta_z^2 + \beta_x^2 - \alpha_z^2 - \alpha_x^2 = \beta^2 - \alpha^2\\ &\beta_z\alpha_z + \beta_x\alpha_x = \beta\alpha\end{aligned} \tag{2.10.5}$$

Because $\boldsymbol{k}$ is orthogonal to both $\hat{\mathbf{y}}$ and $\hat{\mathbf{y}}\times\boldsymbol{k}$, we construct the electric field $\boldsymbol{E}_0$ as the following linear combination of TM and TE terms:

$$\boldsymbol{E}_0 = (\hat{\mathbf{y}}\times\hat{\mathbf{k}})A + \hat{\mathbf{y}}B\,, \qquad \text{where}\quad \hat{\mathbf{k}} = \frac{\boldsymbol{k}}{k_c} = \frac{\boldsymbol{\beta} - j\boldsymbol{\alpha}}{\beta - j\alpha} \tag{2.10.6}$$

This satisfies $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$. Then, the magnetic field becomes:

$$\boldsymbol{H}_0 = \frac{\boldsymbol{k}\times\boldsymbol{E}_0}{\omega\mu} = \frac{1}{\eta_c}[\hat{\mathbf{y}}A - (\hat{\mathbf{y}}\times\hat{\mathbf{k}})B] \tag{2.10.7}$$

The vector $\hat{\mathbf{k}}$ is complex-valued and satisfies $\hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1$. These expressions reduce to Eq. (2.10.2), if $\hat{\mathbf{k}} = \hat{\mathbf{z}}'$.

Waves with a complex $\boldsymbol{k}$ are known as *complex waves*, or *inhomogeneous waves*. In applications, they always appear in connection with some interface between two media. The interface serves either as a reflecting/transmitting surface, or as a guiding surface.

For example, when plane waves are incident obliquely from a lossless dielectric onto a planar interface with a lossy medium, the waves transmitted into the lossy medium are of such complex type. Taking the interface to be the xy-plane and the lossy medium to be the region $z \ge 0$, it turns out that the transmitted waves are characterized by attenuation only in the z-direction. Therefore, Eqs. (2.10.5) apply with $\alpha_z > 0$ and $\alpha_x = 0$. The parameter β_x is fixed by Snel's law, so that Eqs. (2.10.5) provide a system of two equations in the two unknowns β_z and α_z. We discuss this further in Chap. 7.

Wave solutions with complex $\boldsymbol{k} = \boldsymbol{\beta} - j\boldsymbol{\alpha}$ are possible even when the propagation medium is lossless so that $\epsilon_c = \epsilon$ is real, and $\beta = \omega\sqrt{\mu\epsilon}$ and $\alpha = 0$. Then, Eqs. (2.10.4) become $\boldsymbol{\beta}\cdot\boldsymbol{\beta} - \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = \beta^2$ and $\boldsymbol{\beta}\cdot\boldsymbol{\alpha} = 0$. Thus, the constant-amplitude and constant-phase planes are *orthogonal* to each other.

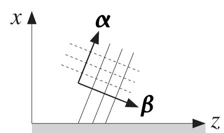
Examples of such waves are the evanescent waves in total internal reflection, various guided-wave problems, such as surface waves, leaky waves, and traveling-wave antennas. The most famous of these is the Zenneck wave, which is a surface wave propagating along a lossy ground, decaying exponentially with distance above and along the ground.

Another example of current interest is surface plasmons [576–614], which are surface waves propagating along the interface between a metal, such as silver, and a dielectric, such as air, with the fields decaying exponentially perpendicularly to the interface both in the air and the metal. We discuss them further in Sections 7.11 and 8.5.

For a classification of various types of complex waves and a review of several applications, including the Zenneck wave, see Refs. [893–900]. We will encounter some of these in Section 7.7.

The table below illustrates the vectorial directions and relative signs of some possible types, assuming that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ lie on the xz plane with the yz plane being the interface plane.

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	α_z	α_x	β_z	β_x	complex wave type
0	↘	0	0	+	−	oblique incidence
↑	→	0	+	+	0	evanescent surface wave
↗	↘	+	+	+	−	Zenneck surface wave
↖	↗	−	+	+	+	leaky wave



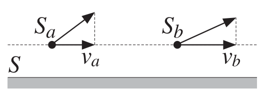
2.11 Doppler Effect

The Doppler effect is the frequency shift perceived by an observer whenever the source of the waves and the observer are in relative motion.

Besides the familiar Doppler effect for sound waves, such as the increase in pitch of the sound of an approaching car, ambulance, or train, the Doppler effect has several other applications, such as Doppler radar for aircraft tracking, weather radar, ground imaging, and police radar; several medical ultrasound applications, such as monitoring blood flow or imaging internal organs and fetuses; and astrophysical applications, such as measuring the red shift of light emitted by receding galaxies.

In the classical treatment of the Doppler effect, one assumes that the waves propagate in some medium (e.g., sound waves in air). If c is the wave propagation speed in the medium, the classical expression for the Doppler effect is given by:

$$\boxed{f_b = f_a\frac{c - v_b}{c - v_a}} \tag{2.11.1}$$



where f_a and f_b are the frequencies measured in the rest frames of the source S_a and observer S_b, and v_a and v_b are the velocities of S_a and S_b with respect to the propagation medium, projected along their line of sight.

The algebraic sign of v_a is positive if S_a is moving toward S_b from the left, and the sign of v_b is positive if S_b is moving away from S_a. Thus, there is a frequency increase whenever the source and the observer are approaching each other ($v_a > 0$ or $v_b < 0$), and a frequency decrease if they are receding from each other ($v_a < 0$ or $v_b > 0$).

Eq. (2.11.1) can be derived by considering the two cases of a moving source and a stationary observer, or a stationary source and a moving observer, as shown in Fig. 2.11.1.

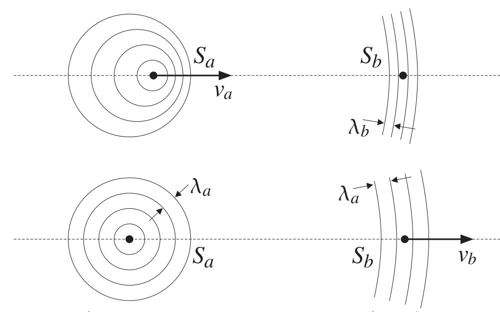


Fig. 2.11.1 Classical Doppler effect.

In the first case, the spacing of the successive crests of the wave (the wavelength) is decreased in front of the source because during the time interval between crests, that is, during one period $T_a = 1/f_a$, the source has moved by a distance v_aT_a bringing two successive crests closer together by that amount. Thus, the wavelength perceived by the observer will be $\lambda_b = \lambda_a - v_aT_a = (c - v_a)/f_a$, which gives:

$$\boxed{f_b = \frac{c}{\lambda_b} = f_a\frac{c}{c - v_a}} \quad \text{(moving source)} \tag{2.11.2}$$

In the second case, because the source is stationary, the wavelength λ_a will not change, but now the effective speed of the wave in the rest frame of the observer is $(c - v_b)$. Therefore, the frequency perceived by the observer will be:

$$\boxed{f_b = \frac{c - v_b}{\lambda_a} = f_a\frac{c - v_b}{c}} \quad \text{(moving observer)} \tag{2.11.3}$$

The combination of these two cases leads to Eq. (2.11.1). We have assumed in Eqs. (2.11.1)–(2.11.3) that v_a, v_b are less than c so that supersonic effects are not considered. A counter-intuitive aspect of the classical Doppler formula (2.11.1) is that it does not depend on the relative velocity $(v_b - v_a)$ of the observer and source. Therefore, it makes a difference whether the source or the observer is moving. Indeed, when the observer is moving with $v_b = v$ away from a stationary source, or when the source is moving with $v_a = -v$ away from a stationary observer, then Eq. (2.11.1) gives:

$$f_b = f_a(1 - v/c)\,, \qquad f_b = \frac{f_a}{1 + v/c} \tag{2.11.4}$$

These two expressions are equivalent to first-order in v/c. This follows from the Taylor series approximation $(1+x)^{-1} \simeq 1-x$, which is valid for $|x| \ll 1$. More generally, to first order in v_a/c and v_b/c, Eq. (2.11.1) does depend only on the relative velocity. In this case the Doppler shift $\Delta f = f_b - f_a$ is given approximately by:

$$\frac{\Delta f}{f_a} = \frac{v_a - v_b}{c} \tag{2.11.5}$$

For Doppler radar this doubles to $\Delta f/f_a = 2(v_a - v_b)/c$ because the wave suffers two Doppler shifts, one for the transmitted and one for the reflected wave. This is further discussed in Sec. 5.8.

For electromagnetic waves,† the correct Doppler formula depends only on the relative velocity between observer and source and is given by the relativistic generalization of Eq. (2.11.1):

$$\boxed{f_b = f_a\sqrt{\frac{c-v_b}{c+v_b}\cdot\frac{c+v_a}{c-v_a}} = f_a\sqrt{\frac{c-v}{c+v}}} \quad \text{(relativistic Doppler effect)} \tag{2.11.6}$$

where v is the velocity of the observer *relative* to the source, which according to the Einstein addition theorem for velocities is given through the equivalent expressions:

$$v = \frac{v_b - v_a}{1 - v_b v_a/c^2} \quad\Leftrightarrow\quad v_b = \frac{v_a + v}{1 + v_a v/c^2} \quad\Leftrightarrow\quad \frac{c-v}{c+v} = \frac{c-v_b}{c+v_b}\cdot\frac{c+v_a}{c-v_a} \tag{2.11.7}$$

Using the first-order Taylor series expansion $(1+x)^{\pm 1/2} = 1 \pm x/2$, one can show that Eq. (2.11.6) can be written approximately as Eq. (2.11.5).

Next, we present a more precise discussion of the Doppler effect based on Lorentz transformations. Our discussion follows that of Einstein's 1905 paper on special relativity [458]. Fig. 2.11.2 shows a uniform plane wave propagating in vacuum as viewed from the vantage point of two coordinate frames: a fixed frame S and a frame S' moving towards the z-direction with velocity v. We assume that the wavevector $\mathbf{k}$ in S lies in the xz-plane and forms an angle θ with the z-axis as shown.

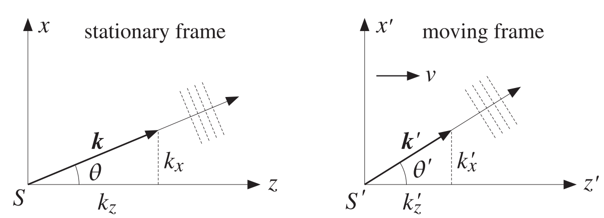


Fig. 2.11.2 Plane wave viewed from stationary and moving frames.

As discussed in Appendix H, the transformation of the frequency-wavenumber four-vector $(\omega/c, \mathbf{k})$ between the frames S and S' is given by the Lorentz transformation of

†The question of the existence of a medium (the ether) required for the propagation of electromagnetic waves precipitated the development of the special relativity theory.

Eq. (H.14). Because $k_y = 0$ and the transverse components of $\mathbf{k}$ do not change, we will have $k'_y = k_y = 0$, that is, the wavevector $\mathbf{k}'$ will still lie in the xz-plane of the S' frame. The frequency and the other components of $\mathbf{k}$ transform as follows:

$$\boxed{\begin{aligned} \omega' &= \gamma(\omega - \beta c k_z) \\ k'_z &= \gamma\Big(k_z - \frac{\beta}{c}\omega\Big) \\ k'_x &= k_x \end{aligned}} \qquad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.11.8}$$

Setting $k_z = k\cos\theta$, $k_x = k\sin\theta$, with $k = \omega/c$, and similarly in the S' frame, $k'_z = k'\cos\theta'$, $k'_x = k'\sin\theta'$, with $k' = \omega'/c$, Eqs. (2.11.8) may be rewritten in the form:

$$\begin{aligned} \omega' &= \omega\gamma(1 - \beta\cos\theta) \\ \omega'\cos\theta' &= \omega\gamma(\cos\theta - \beta) \\ \omega'\sin\theta' &= \omega\sin\theta \end{aligned} \tag{2.11.9}$$

The first equation is the relativistic Doppler formula, relating the frequency of the wave as it is measured by an observer in the moving frame S' to the frequency of a source in the fixed frame S:

$$\boxed{f' = f\gamma(1 - \beta\cos\theta) = f\frac{1 - \beta\cos\theta}{\sqrt{1-\beta^2}}} \tag{2.11.10}$$

The last two equations in (2.11.9) relate the apparent propagation angles θ, θ' in the two frames. Eliminating ω, ω', we obtain the following equivalent expressions:

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \;\Leftrightarrow\; \sin\theta' = \frac{\sin\theta}{\gamma(1 - \beta\cos\theta)} \;\Leftrightarrow\; \frac{\tan(\theta'/2)}{\tan(\theta/2)} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{2.11.11}$$

where to obtain the last one we used the identity $\tan(\phi/2) = \sin\phi/(1+\cos\phi)$. The difference in the propagation angles θ, θ' is referred to as the *aberration* of light due to motion. Using Eqs. (2.11.11), the Doppler equation (2.11.10) may be written in the alternative forms:

$$f' = f\gamma(1 - \beta\cos\theta) = \frac{f}{\gamma(1 + \beta\cos\theta')} = f\sqrt{\frac{1 - \beta\cos\theta}{1 + \beta\cos\theta'}} \tag{2.11.12}$$

If the wave is propagating in the z-direction ($\theta = 0^o$), Eq. (2.11.10) gives:

$$f' = f\sqrt{\frac{1-\beta}{1+\beta}} \tag{2.11.13}$$

and, if it is propagating in the x-direction ($\theta = 90^o$), we obtain the so-called *transverse* Doppler effect: $f' = f\gamma$. The relativistic Doppler effect, including the transverse one, has been observed experimentally.

To derive Eq. (2.11.6), we consider two reference frames S_a, S_b moving along the z-direction with velocities v_a, v_b with respect to our fixed frame S, and we assume that

$\theta = 0^{\circ}$ in the frame S. Let f_a, f_b be the frequencies of the wave as measured in the frames S_a, S_b. Then, the separate application of Eq. (2.11.13) to S_a and S_b gives:

$$f_a = f\sqrt{\frac{1-\beta_a}{1+\beta_a}}\,, \quad f_b = f\sqrt{\frac{1-\beta_b}{1+\beta_b}} \quad \Rightarrow \quad f_b = f_a\sqrt{\frac{1-\beta_b}{1+\beta_b}\cdot\frac{1+\beta_a}{1-\beta_a}} \tag{2.11.14}$$

where $\beta_a = v_a/c$ and $\beta_b = v_b/c$. This is equivalent to Eq. (2.11.6). The case when the wave is propagating in an arbitrary direction θ is given in Problem 2.27.

Next, we consider the transformation of the electromagnetic field components between the two frames. The electric field has the following form in S and S':

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{j(\omega t - k_x x - k_z z)}\,, \quad \boldsymbol{E}' = \boldsymbol{E}_0' e^{j(\omega' t' - k_x' x' - k_z' z')} \tag{2.11.15}$$

As we discussed in Appendix H, the propagation phase factors remain invariant in the two frames, that is, $\omega t - k_x x - k_z z = \omega' t' - k_x' x' - k_z' z'$. Assuming a TE wave and using Eq. (2.9.9), the electric and magnetic field amplitudes will have the following form in the two frames:

$$\begin{aligned} \boldsymbol{E}_0 &= E_0 \hat{\mathbf{y}}\,, \quad c\boldsymbol{B}_0 = \eta_0 \boldsymbol{H}_0 = \hat{\mathbf{k}} \times \boldsymbol{E}_0 = E_0(-\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \\ \boldsymbol{E}_0' &= E_0' \hat{\mathbf{y}}\,, \quad c\boldsymbol{B}_0' = \eta_0 \boldsymbol{H}_0' = \hat{\mathbf{k}}' \times \boldsymbol{E}_0' = E_0'(-\hat{\mathbf{x}}\cos\theta' + \hat{\mathbf{z}}\sin\theta') \end{aligned} \tag{2.11.16}$$

Applying the Lorentz transformation properties of Eq. (H.31) to the above field components, we find:

$$\begin{aligned} E_y' &= \gamma(E_y + \beta c B_x) & & E_0' = E_0\gamma(1 - \beta\cos\theta) \\ cB_x' &= \gamma(cB_x + \beta E_y) \quad \Rightarrow & & -E_0'\cos\theta' = -E_0\gamma(\cos\theta - \beta) \\ cB_z' &= cB_z & & E_0'\sin\theta' = E_0\sin\theta \end{aligned} \tag{2.11.17}$$

The first equation gives the desired relationship between E_0 and E_0'. The last two equations imply the same angle relationships as Eq. (2.11.11). The same relationship between E_0, E_0' holds also for a TM wave defined by $\boldsymbol{E}_0 = E_0(\hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta)$.

2.12 Propagation in Negative-Index Media

In media with simultaneously negative permittivity and permeability, $\epsilon < 0$ and $\mu < 0$, the refractive index must be negative [376]. To see this, we consider a uniform plane wave propagating in a lossless medium:

$$E_x(z,t) = E_0\, e^{j\omega t - jkz}\,, \quad H_y(z,t) = H_0\, e^{j\omega t - jkz}$$

Then, Maxwell's equations require the following relationships, which are equivalent to Faraday's and Ampère's laws, respectively:

$$kE_0 = \omega\mu H_0\,, \quad kH_0 = \omega\epsilon E_0\,, \quad \text{or,}$$

$$\eta = \frac{E_0}{H_0} = \frac{\omega\mu}{k} = \frac{k}{\omega\epsilon} \quad \Rightarrow \quad k^2 = \omega^2\epsilon\mu \tag{2.12.1}$$

Because the medium is lossless, k and η will be real and the time-averaged Poynting vector, which points in the z-direction, will be:

$$\mathcal{P}_z = \frac{1}{2}\,\text{Re}[E_0 H_0^*] = \frac{1}{2\eta}|E_0|^2 = \frac{1}{2}\,\eta|H_0|^2 \tag{2.12.2}$$

If we require that the energy flux is towards the positive z-direction, that is, $\mathcal{P}_z > 0$, then we must have $\eta > 0$. Because μ and ϵ are negative, Eq. (2.12.1) implies that k must be negative, $k < 0$, in order for the ratio $\eta = \omega\mu/k$ to be positive. Thus, in solving $k^2 = \omega^2\mu\epsilon$, we must choose the negative square root:

$$k = -\omega\sqrt{\mu\epsilon} \tag{2.12.3}$$

The refractive index n may be defined through $k = k_0 n$, where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the free-space wavenumber. Thus, we have $n = k/k_0 = -\sqrt{\mu\epsilon/\mu_0\epsilon_0} = -\sqrt{\mu_{\text{rel}}\epsilon_{\text{rel}}}$, expressed in terms of the relative permittivity and permeability. Writing $\epsilon = -|\epsilon|$ and $\mu = -|\mu|$, we have for the medium impedance:

$$\eta = \frac{\omega\mu}{k} = \frac{-\omega|\mu|}{-\omega\sqrt{|\mu\epsilon|}} = \sqrt{\frac{|\mu|}{|\epsilon|}} = \sqrt{\frac{\mu}{\epsilon}} \tag{2.12.4}$$

which can be written also as follows, where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$:

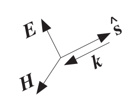
$$\eta = \eta_0\,\frac{\mu}{\mu_0 n} = \eta_0\,\frac{\epsilon_0 n}{\epsilon} \tag{2.12.5}$$

Thus, in negative-index media, the wave vector k and the phase velocity $v_{\text{ph}} = \omega/k = c_0/n$ will be negative, pointing in opposite direction than the Poynting vector. As we saw in Sec. 1.18, for lossless negative-index media the energy transport velocity v_{en}, which is in the direction of the Poynting vector, coincides with the group velocity v_g. Thus, $v_g = v_{\text{en}} > 0$, while $v_{\text{ph}} < 0$.

Two consequences of the negative refractive index, $n < 0$, are the reversal of Snel's law discussed in Sec. 7.16 and the possibility of a perfect lens discussed in Sec. 8.6. These and other consequences of $n < 0$, such as the reversal of the Doppler and Cherenkov effects and the reversal of the field momentum, have been discussed by Veselago [376].

If the propagation is along an arbitrary direction defined by a unit-vector $\hat{\mathbf{s}}$ (i.e., a rotated version of $\hat{\mathbf{z}}$), then we may define the wavevector by $\boldsymbol{k} = k\hat{\mathbf{s}}$, with k to be determined, and look for solutions of Maxwell's equations of the form:

$$\begin{aligned} \boldsymbol{E}(\mathbf{r},t) &= \boldsymbol{E}_0\, e^{j\omega t - j\,\boldsymbol{k}\cdot\mathbf{r}} \\ \boldsymbol{H}(\mathbf{r},t) &= \boldsymbol{H}_0\, e^{j\omega t - j\,\boldsymbol{k}\cdot\mathbf{r}} \end{aligned} \tag{2.12.6}$$



Gauss's laws require that the constant vectors $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ be transverse to $\boldsymbol{k}$, or $\hat{\mathbf{s}}$, that is, $\hat{\mathbf{s}}\cdot\boldsymbol{E}_0 = \hat{\mathbf{s}}\cdot\boldsymbol{H}_0 = 0$. Then, Faraday's and Ampère's laws require that:

$$\boldsymbol{H}_0 = \frac{1}{\eta}\,(\hat{\mathbf{s}}\times\boldsymbol{E}_0)\,, \quad \eta = \frac{\omega\mu}{k} = \frac{k}{\omega\epsilon} \quad \Rightarrow \quad k^2 = \omega^2\mu\epsilon \tag{2.12.7}$$

with a Poynting vector:

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\,\text{Re}\big[\boldsymbol{E}_0\times\boldsymbol{H}_0^*\big] = \hat{\mathbf{s}}\,\frac{1}{2\eta}\,|\boldsymbol{E}_0|^2 \tag{2.12.8}$$

Thus, if $\boldsymbol{\mathcal{P}}$ is assumed to be in the direction of $\hat{\mathbf{s}}$, then we must have $\eta > 0$, and therefore, k must be negative as in Eq. (2.12.3). It follows that the wavevector $\boldsymbol{k} = k\hat{\mathbf{s}}$ will be in the opposite direction of $\hat{\mathbf{s}}$ and $\boldsymbol{\mathcal{P}}$. Eq. (2.12.7) implies that the triplet $\{\boldsymbol{E}_0, \boldsymbol{H}_0, \hat{\mathbf{s}}\}$ is still a right-handed vector system, but $\{\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{k}\}$ will be a *left-handed* system. This is the reason why Veselago [376] named such media *left-handed media.*[†]

In a lossy negative-index medium, the permittivity and permeability will be complex-valued, $\epsilon = \epsilon_r - j\epsilon_i$ and $\mu = \mu_r - j\mu_i$, with negative real parts $\epsilon_r, \mu_r < 0$, and positive imaginary parts $\epsilon_i, \mu_i > 0$. Eq. (2.12.1) remains the same and will imply that k and η will be complex-valued. Letting $k = \beta - j\alpha$, the fields will be attenuating as they propagate:

$$E_x(z,t) = E_0 e^{-\alpha z} e^{j\omega t - j\beta z}\,, \quad H_y(z,t) = H_0 e^{-\alpha z} e^{j\omega t - j\beta z}$$

and the Poynting vector will be given by:

$$\mathcal{P}_z = \frac{1}{2}\mathrm{Re}\big[E_x(z)H_y^*(z)\big] = \frac{1}{2}\mathrm{Re}\big(\frac{1}{\eta}\big)|E_0|^2 e^{-2\alpha z} = \frac{1}{2}\mathrm{Re}(\eta)|H_0|^2 e^{-2\alpha z} \tag{2.12.9}$$

The refractive index is complex-valued, $n = n_r - jn_i$, and is related to k through $k = k_0 n$, or, $\beta - j\alpha = k_0(n_r - jn_i)$, or, $\beta = k_0 n_r$ and $\alpha = k_0 n_i$. Thus, the conditions of negative phase velocity ($\beta < 0$), field attenuation ($\alpha > 0$), and positive power flow can be stated equivalently as follows:

$$n_r < 0\,, \quad n_i > 0\,, \quad \mathrm{Re}(\eta) > 0 \tag{2.12.10}$$

Next, we look at the necessary and sufficient conditions for a medium to satisfy these conditions. If we express ϵ, μ in their polar forms, $\epsilon = |\epsilon|e^{-j\theta_\epsilon}$ and $\mu = |\mu|e^{-j\theta_\mu}$, then, regardless of the signs of ϵ_r, μ_r, the assumption that the medium is lossy, $\epsilon_i, \mu_i > 0$, requires that $\sin\theta_\epsilon > 0$ and $\sin\theta_\mu > 0$, and these are equivalent to the restrictions:

$$0 \le \theta_\epsilon \le \pi\,, \quad 0 \le \theta_\mu \le \pi \tag{2.12.11}$$

To guarantee $\alpha > 0$, the wavenumber k must be computed by taking the positive square root of $k^2 = \omega^2\mu\epsilon = \omega^2|\mu\epsilon|^2 e^{-j(\theta_\epsilon + \theta_\mu)}$, that is,

$$k = \beta - j\alpha = \omega\sqrt{|\mu\epsilon|}\, e^{-j\theta_+}\,, \quad \theta_+ = \frac{\theta_\epsilon + \theta_\mu}{2} \tag{2.12.12}$$

Indeed, the restrictions (2.12.11) imply the same for θ_+, that is, $0 \le \theta_+ \le \pi$, or, equivalently, $\sin\theta_+ > 0$, and hence $\alpha > 0$. Similarly, the quantities n, η are given by:

$$n = |n|\, e^{-j\theta_+}\,, \quad \eta = |\eta|\, e^{-j\theta_-}\,, \quad \theta_- = \frac{\theta_\epsilon - \theta_\mu}{2} \tag{2.12.13}$$

where $|n| = \sqrt{|\mu\epsilon|/\mu_0\epsilon_0}$ and $|\eta| = \sqrt{|\mu|/|\epsilon|}$. It follows that $n_i = |n|\sin\theta_+ > 0$. Since $n_r = |n|\cos\theta_+$ and $\mathrm{Re}(\eta) = |\eta|\cos\theta_-$, the conditions $n_r < 0$ and $\mathrm{Re}(\eta) > 0$ will be equivalent to

$$\cos\theta_+ = \cos\Big(\frac{\theta_\epsilon + \theta_\mu}{2}\Big) < 0\,, \quad \cos\theta_- = \cos\Big(\frac{\theta_\epsilon - \theta_\mu}{2}\Big) > 0 \tag{2.12.14}$$

[†]The term negative-index media is preferred in order to avoid confusion with chiral media.

Using some trigonometric identities, these conditions become equivalently:

$$\cos(\theta_\epsilon/2)\cos(\theta_\mu/2) - \sin(\theta_\epsilon/2)\sin(\theta_\mu/2) < 0$$
$$\cos(\theta_\epsilon/2)\cos(\theta_\mu/2) + \sin(\theta_\epsilon/2)\sin(\theta_\mu/2) > 0$$

which combine into

$$-\sin(\theta_\epsilon/2)\sin(\theta_\mu/2) < \cos(\theta_\epsilon/2)\cos(\theta_\mu/2) < \sin(\theta_\epsilon/2)\sin(\theta_\mu/2)$$

Because $0 \le \theta_\epsilon/2 \le \pi/2$, we have $\cos(\theta_\epsilon/2) \ge 0$ and $\sin(\theta_\epsilon/2) \ge 0$, and similarly for $\theta_\mu/2$. Thus, the above conditions can be replaced by the single equivalent inequality:

$$\tan(\theta_\epsilon/2)\tan(\theta_\mu/2) > 1 \tag{2.12.15}$$

A number of equivalent conditions have been given in the literature [397,425] for a medium to have negative phase velocity and positive power:

$$\begin{aligned}(|\epsilon| - \epsilon_r)\,(|\mu| - \mu_r) &> \epsilon_i\mu_i\\ \epsilon_r|\mu| + \mu_r|\epsilon| &< 0\\ \epsilon_r\mu_i + \mu_r\epsilon_i &< 0\end{aligned} \tag{2.12.16}$$

They are all equivalent to condition (2.12.15). This can be seen by writing them in terms of the angles $\theta_\epsilon, \theta_\mu$ and then using simple trigonometric identities, such as $\tan(\theta/2) = (1 - \cos\theta)/\sin\theta$, to show their equivalence to (2.12.15):

$$\begin{aligned}(1 - \cos\theta_\epsilon)\,(1 - \cos\theta_\mu) &> \sin\theta_\epsilon\sin\theta_\mu\\ \cos\theta_\epsilon + \cos\theta_\mu &< 0\\ \cot\theta_\epsilon + \cot\theta_\mu &< 0\end{aligned} \tag{2.12.17}$$

If the medium has negative real parts, $\epsilon_r < 0$ and $\mu_r < 0$, then the conditions (2.12.16) are obviously satisfied.

2.13 Problems

2.1 A function $E(z,t)$ may be thought of as a function $E(\zeta, \xi)$ of the independent variables $\zeta = z - ct$ and $\xi = z + ct$. Show that the wave equation (2.1.6) and the forward-backward equations (2.1.10) become in these variables:

$$\frac{\partial^2 \boldsymbol{E}}{\partial\zeta\partial\xi} = 0\,, \qquad \frac{\partial \boldsymbol{E}_+}{\partial\xi} = 0\,, \qquad \frac{\partial \boldsymbol{E}_-}{\partial\zeta} = 0$$

Thus, $\boldsymbol{E}_+$ may depend only on ζ and $\boldsymbol{E}_-$ only on ξ.

2.2 A source located at $z = 0$ generates an electromagnetic pulse of duration of T sec, given by $\boldsymbol{E}(0,t) = \hat{\mathbf{x}}E_0\big[u(t) - u(t - T)\big]$, where $u(t)$ is the unit step function and E_0 is a constant. The pulse is launched towards the positive z-direction. Determine expressions for $\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(z,t)$ and sketch them versus z at any given t.

2.3 Show that for a single-frequency wave propagating along the z-direction the corresponding transverse fields $\mathbf{E}(z),\mathbf{H}(z)$ satisfy the system of equations:

$$\frac{\partial}{\partial z}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\times\hat{\mathbf{z}}\end{bmatrix}=\begin{bmatrix}0 & -j\omega\mu\\ -j\omega\epsilon & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\times\hat{\mathbf{z}}\end{bmatrix}$$

where the matrix equation is meant to apply individually to the x,y components of the vector entries. Show that the following similarity transformation diagonalizes the transition matrix, and discuss its role in decoupling and solving the above system in terms of forward and backward waves:

$$\begin{bmatrix}1 & \eta\\ 1 & -\eta\end{bmatrix}\begin{bmatrix}0 & -j\omega\mu\\ -j\omega\epsilon & 0\end{bmatrix}\begin{bmatrix}1 & \eta\\ 1 & -\eta\end{bmatrix}^{-1}=\begin{bmatrix}-jk & 0\\ 0 & jk\end{bmatrix}$$

where $k=\omega/c$, $c=1/\sqrt{\mu\epsilon}$, and $\eta=\sqrt{\mu/\epsilon}$.

2.4 The visible spectrum has the wavelength range 380–780 nm. What is this range in THz? In particular, determine the frequencies of red, orange, yellow, green, blue, and violet having the nominal wavelengths of 700, 610, 590, 530, 470, and 420 nm.

2.5 What is the frequency in THz of a typical CO_2 laser (used in laser surgery) having the far infrared wavelength of 20 μm?

2.6 What is the wavelength in meters or cm of a wave with the frequencies of 10 kHz, 10 MHz, and 10 GHz?

What is the frequency in GHz of the 21-cm hydrogen line observed in the cosmos?

What is the wavelength in cm of the typical microwave oven frequency of 2.45 GHz?

2.7 Suppose you start with $\mathbf{E}(z,t)=\hat{\mathbf{x}}E_0e^{j\omega t-jkz}$, but you do not yet know the relationship between k and ω (you may assume they are both positive.) By inserting $\mathbf{E}(z,t)$ into Maxwell's equations, determine the k–ω relationship as a consequence of these equations. Determine also the magnetic field $\mathbf{H}(z,t)$ and verify that all of Maxwell's equations are satisfied.

Repeat the problem if $\mathbf{E}(z,t)=\hat{\mathbf{x}}E_0e^{j\omega t+jkz}$ and if $\mathbf{E}(z,t)=\hat{\mathbf{y}}E_0e^{j\omega t-jkz}$.

2.8 Determine the polarization types of the following waves, and indicate the direction, if linear, and the sense of rotation, if circular or elliptic:

a. $\mathbf{E}=E_0(\hat{\mathbf{x}}+\hat{\mathbf{y}})e^{-jkz}$ e. $\mathbf{E}=E_0(\hat{\mathbf{x}}-\hat{\mathbf{y}})e^{-jkz}$

b. $\mathbf{E}=E_0(\hat{\mathbf{x}}-\sqrt{3}\,\hat{\mathbf{y}})e^{-jkz}$ f. $\mathbf{E}=E_0(\sqrt{3}\,\hat{\mathbf{x}}-\hat{\mathbf{y}})e^{-jkz}$

c. $\mathbf{E}=E_0(j\hat{\mathbf{x}}+\hat{\mathbf{y}})e^{-jkz}$ g. $\mathbf{E}=E_0(j\hat{\mathbf{x}}-\hat{\mathbf{y}})e^{jkz}$

d. $\mathbf{E}=E_0(\hat{\mathbf{x}}-2j\,\hat{\mathbf{y}})e^{-jkz}$ h. $\mathbf{E}=E_0(\hat{\mathbf{x}}+2j\,\hat{\mathbf{y}})e^{jkz}$

2.9 A uniform plane wave, propagating in the z-direction in vacuum, has the following electric field:

$$\boldsymbol{\mathcal{E}}(t,z)=2\,\hat{\mathbf{x}}\,\cos(\omega t-kz)+4\,\hat{\mathbf{y}}\,\sin(\omega t-kz)$$

a. Determine the vector phasor representing $\boldsymbol{\mathcal{E}}(t,z)$ in the complex form $\mathbf{E}=\mathbf{E}_0e^{j\omega t-jkz}$.

b. Determine the polarization of this electric field (linear, circular, elliptic, left-handed, right-handed?)

c. Determine the magnetic field $\boldsymbol{\mathcal{H}}(t,z)$ in its real-valued form.

2.10 A uniform plane wave propagating in vacuum along the z direction has real-valued electric field components:

$$\mathcal{E}_x(z,t)=\cos(\omega t-kz)\,,\quad \mathcal{E}_y(z,t)=2\sin(\omega t-kz)$$

a. Its phasor form has the form $\mathbf{E}=(A\,\hat{\mathbf{x}}+B\,\hat{\mathbf{y}})e^{\pm jkz}$. Determine the numerical values of the complex-valued coefficients A,B and the correct sign of the exponent.

b. Determine the polarization of this wave (left, right, linear, etc.). Explain your reasoning.

2.11 Consider the two electric fields, one given in its real-valued form, and the other, in its phasor form:

a. $\boldsymbol{\mathcal{E}}(t,z)=\hat{\mathbf{x}}\sin(\omega t+kz)+2\hat{\mathbf{y}}\cos(\omega t+kz)$

b. $\mathbf{E}(z)=\big[(1+j)\hat{\mathbf{x}}-(1-j)\hat{\mathbf{y}}\big]e^{-jkz}$

For both cases, determine the polarization of the wave (linear, circular, left, right, etc.) and the direction of propagation.

For case (a), determine the field in its phasor form. For case (b), determine the field in its real-valued form as a function of t,z.

2.12 A uniform plane wave propagating in the z-direction has the following real-valued electric field:

$$\boldsymbol{\mathcal{E}}(t,z)=\hat{\mathbf{x}}\cos(\omega t-kz-\pi/4)+\hat{\mathbf{y}}\cos(\omega t-kz+\pi/4)$$

a. Determine the complex-phasor form of this electric field.

b. Determine the corresponding magnetic field $\boldsymbol{\mathcal{H}}(t,z)$ given in its real-valued form.

c. Determine the polarization type (left, right, linear, etc.) of this wave.

2.13 Determine the polarization type (left, right, linear, etc.) and the direction of propagation of the following electric fields given in their phasor forms:

a. $\mathbf{E}(z)=\big[(1+j\sqrt{3})\hat{\mathbf{x}}+2\,\hat{\mathbf{y}}\big]e^{+jkz}$

b. $\mathbf{E}(z)=\big[(1+j)\hat{\mathbf{x}}-(1-j)\hat{\mathbf{y}}\big]e^{-jkz}$

c. $\mathbf{E}(z)=\big[\hat{\mathbf{x}}-\hat{\mathbf{z}}+j\sqrt{2}\,\hat{\mathbf{y}}\big]e^{-jk(x+z)/\sqrt{2}}$

2.14 Consider a forward-moving wave in its real-valued form:

$$\boldsymbol{\mathcal{E}}(t,z)=\hat{\mathbf{x}}A\cos(\omega t-kz+\phi_a)+\hat{\mathbf{y}}B\cos(\omega t-kz+\phi_b)$$

Show that:

$$\boldsymbol{\mathcal{E}}(t+\Delta t,z+\Delta z)\times\boldsymbol{\mathcal{E}}(t,z)=\hat{\mathbf{z}}AB\sin(\phi_a-\phi_b)\sin(\omega\Delta t-k\Delta z)$$

2.15 In this problem we explore the properties of the polarization ellipse. Let us assume initially that $A\neq B$. Show that in order for the polarization ellipse of Eq. (2.5.4) to be equivalent to the rotated one of Eq. (2.5.8), we must determine the tilt angle θ to satisfy the following matrix condition:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\dfrac{1}{A^2} & -\dfrac{\cos\phi}{AB}\\ -\dfrac{\cos\phi}{AB} & \dfrac{1}{B^2}\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\sin^2\phi\begin{bmatrix}\dfrac{1}{A'^2} & 0\\ 0 & \dfrac{1}{B'^2}\end{bmatrix}\qquad(2.13.1)$$

From this condition, show that θ must satisfy Eq. (2.5.5). However, this equation does not determine θ uniquely. To see this, let $\tau=\tan\theta$ and use a standard trigonometric identity to write (2.5.5) in the form:

$$\tan 2\theta=\frac{2\tau}{1-\tau^2}=\frac{2AB}{A^2-B^2}\cos\phi\qquad(2.13.2)$$

Show that the two possible solutions for τ are given by:

$$\tau_s = \frac{B^2 - A^2 + sD}{2AB\cos\phi}, \quad s = \pm 1$$

where

$$D = \sqrt{(A^2 - B^2)^2 + 4A^2B^2\cos^2\phi} = \sqrt{(A^2 + B^2)^2 - 4A^2B^2\sin^2\phi}$$

Show also that $\tau_s\tau_{-s} = -1$. Thus one or the other of the τ's must have magnitude less than unity. To determine which one, show the relationship:

$$1 - \tau_s^2 = \frac{s(A^2 - B^2)\big[D - s(A^2 - B^2)\big]}{2A^2B^2\cos^2\phi}$$

Show that the quantity $D - s(A^2 - B^2)$ is always positive. If we select $s = \text{sign}(A - B)$, then $s(A^2 - B^2) = |A^2 - B^2|$, and therefore, $1 - \tau_s^2 > 0$, or $|\tau_s| < 1$. This is the proper choice of τ_s and corresponding tilt angle θ. We note parenthetically, that if Eq. (2.13.2) is solved by taking arc tangents of both sides,

$$\theta = \frac{1}{2}\text{atan}\left[\frac{2AB}{A^2 - B^2}\cos\phi\right] \tag{2.13.3}$$

then, because MATLAB constrains the returned angle from the arctan function to lie in the interval $-\pi/2 \le 2\theta \le \pi/2$, it follows that θ will lie in $-\pi/4 \le \theta \le \pi/4$, which always results in a tangent with $|\tan\theta| \le 1$. Thus, (2.13.3) generates the proper θ corresponding to τ_s with $s = \text{sign}(A - B)$. In fact, our function `ellipse` uses (2.13.3) internally. The above results can be related to the eigenvalue properties of the matrix,

$$Q = \begin{bmatrix} \dfrac{1}{A^2} & -\dfrac{\cos\phi}{AB} \\ -\dfrac{\cos\phi}{AB} & \dfrac{1}{B^2} \end{bmatrix}$$

defined by the quadratic form of the polarization ellipse (2.5.4). Show that Eq. (2.13.1) is equivalent to the eigenvalue decomposition of Q, with the diagonal matrix on the right-hand side representing the two eigenvalues, and $[\cos\theta, \sin\theta]^T$ and $[-\sin\theta, \cos\theta]^T$, the corresponding eigenvectors. By solving the characteristic equation $\det(Q - \lambda I) = 0$, show that the two eigenvalues of Q are given by:

$$\lambda_s = \frac{A^2 + B^2 + sD}{2A^2B^2}, \quad s = \pm 1$$

Thus, it follows from (2.13.1) that $\sin^2\phi/A'^2$ and $\sin^2\phi/B'^2$ must be identified with one or the other of the two eigenvalues λ_s, λ_{-s}. From Eq. (2.13.1) show the relationships:

$$\lambda_s\lambda_{-s} = \frac{\sin^2\phi}{A^2B^2}, \quad \frac{1}{A^2} - \frac{\cos\phi}{AB}\tau_s = \lambda_{-s}, \quad \frac{1}{B^2} + \frac{\cos\phi}{AB}\tau_s = \lambda_s$$

With the choice $s = \text{sign}(A - B)$, show that the ellipse semi-axes are given by the following equivalent expressions:

$$\begin{aligned} A'^2 &= A^2 + \tau_s AB\cos\phi = \frac{A^2 - B^2\tau_s^2}{1 - \tau_s^2} = \frac{1}{2}\big[A^2 + B^2 + sD\big] = A^2B^2\lambda_s \\ B'^2 &= B^2 - \tau_s AB\cos\phi = \frac{B^2 - A^2\tau_s^2}{1 - \tau_s^2} = \frac{1}{2}\big[A^2 + B^2 - sD\big] = A^2B^2\lambda_{-s} \end{aligned} \tag{2.13.4}$$

with the right-most expressions being equivalent to Eqs. (2.5.6). Show also the following:

$$A'^2 + B'^2 = A^2 + B^2, \quad A'B' = AB|\sin\phi|$$

Using these relationships and the definition (2.5.9) for the angle χ, show that $\tan\chi$ is equal to the minor-to-major axis ratio B'/A' or A'/B', whichever is less than one.

Finally, for the special case $A = B$, by directly manipulating the polarization ellipse (2.5.4), show that $\theta = \pi/4$ and that A', B' are given by Eq. (2.5.10). Since $\tau = 1$ in this case, the left-most equations in (2.13.4) generate the same A', B'. Show that one can also choose $\theta = -\pi/4$ or $\tau = -1$, with A', B' reversing roles, but with the polarization ellipse remaining the same.

2.16 Show the cross-product equation (2.5.11). Then, prove the more general relationship:

$$\boldsymbol{\mathcal{E}}(t_1)\times\boldsymbol{\mathcal{E}}(t_2) = \hat{\mathbf{z}}\,AB\sin\phi\sin\big(\omega(t_2 - t_1)\big)$$

Discuss how linear polarization can be explained with the help of this result.

2.17 Using the properties $k_c\eta_c = \omega\mu$ and $k_c^2 = \omega^2\mu\epsilon_c$ for the complex-valued quantities k_c, η_c of Eq. (2.6.5), show the following relationships, where $\epsilon_c = \epsilon' - j\epsilon''$ and $k_c = \beta - j\alpha$:

$$\text{Re}(\eta_c^{-1}) = \frac{\omega\epsilon''}{2\alpha} = \frac{\beta}{\omega\mu}$$

2.18 Show that for a lossy medium the complex-valued quantities k_c and η_c may be expressed as follows, in terms of the loss angle θ defined in Eq. (2.6.27):

$$\begin{aligned} k_c &= \beta - j\alpha = \omega\sqrt{\mu\epsilon_d'}\left(\cos\frac{\theta}{2} - j\sin\frac{\theta}{2}\right)(\cos\theta)^{-1/2} \\ \eta_c &= \eta' + j\eta'' = \sqrt{\frac{\mu}{\epsilon_d'}}\left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right)(\cos\theta)^{1/2} \end{aligned}$$

2.19 It is desired to reheat frozen mashed potatoes and frozen cooked carrots in a microwave oven operating at 2.45 GHz. Determine the penetration depth and assess the effectiveness of this heating method. Moreover, determine the attenuation of the electric field (in dB and absolute units) at a depth of 1 cm from the surface of the food. The complex dielectric constants of the mashed potatoes and carrots are (see [137]) $\epsilon_c = (65 - j25)\epsilon_0$ and $\epsilon_c = (75 - j25)\epsilon_0$.

2.20 We wish to shield a piece of equipment from RF interference over the frequency range from 10 kHz to 1 GHz by enclosing it in a copper enclosure. The RF interference inside the enclosure is required to be at least 50 dB down compared to its value outside. What is the minimum thickness of the copper shield in mm?

2.21 In order to protect a piece of equipment from RF interference, we construct an enclosure made of aluminum foil (you may assume a reasonable value for its thickness.) The conductivity of aluminum is 3.5×10^7 S/m. Over what frequency range can this shield protect our equipment assuming the same 50-dB attenuation requirement of the previous problem?

2.22 A uniform plane wave propagating towards the positive z-direction in empty space has an electric field at $z = 0$ that is a linear superposition of two components of frequencies ω_1 and ω_2:

$$\mathbf{E}(0, t) = \hat{\mathbf{x}}\,(E_1 e^{j\omega_1 t} + E_2 e^{j\omega_2 t})$$

Determine the fields $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ at any point z.

2.23 An electromagnetic wave propagating in a lossless dielectric is described by the electric and magnetic fields, $\boldsymbol{E}(z)=\hat{\mathbf{x}}E(z)$ and $\boldsymbol{H}(z)=\hat{\mathbf{y}}H(z)$, consisting of the forward and backward components:

$$E(z) = E_+e^{-jkz} + E_-e^{jkz}$$
$$H(z) = \frac{1}{\eta}(E_+e^{-jkz} - E_-e^{jkz})$$

a. Verify that these expressions satisfy all of Maxwell's equations.

b. Show that the time-averaged energy flux in the z-direction is independent of z and is given by:

$$\mathcal{P}_z = \frac{1}{2}\mathrm{Re}\big[E(z)H^*(z)\big] = \frac{1}{2\eta}\big(|E_+|^2 - |E_-|^2\big)$$

c. Assuming $\mu=\mu_0$ and $\epsilon = n^2\epsilon_0$, so that n is the refractive index of the dielectric, show that the fields at two different z-locations, say at $z=z_1$ and $z=z_2$ are related by the matrix equation:

$$\begin{bmatrix} E(z_1) \\ \eta_0 H(z_1)\end{bmatrix} = \begin{bmatrix}\cos kl & jn^{-1}\sin kl \\ jn\sin kl & \cos kl\end{bmatrix}\begin{bmatrix} E(z_2) \\ \eta_0 H(z_2)\end{bmatrix}$$

where $l = z_2 - z_1$, and we multiplied the magnetic field by $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ in order to give it the same dimensions as the electric field.

d. Let $Z(z)=\dfrac{E(z)}{\eta_0 H(z)}$ and $Y(z)=\dfrac{1}{Z(z)}$ be the *normalized* wave impedance and admittance at location z. Show the relationships at at the locations z_1 and z_2:

$$Z(z_1)=\frac{Z(z_2)+jn^{-1}\tan kl}{1+jnZ(z_2)\tan kl}\,,\qquad Y(z_1)=\frac{Y(z_2)+jn\tan kl}{1+jn^{-1}Y(z_2)\tan kl}$$

What would be these relationships if had we normalized to the medium impedance, that is, $Z(z)=E(z)/\eta H(z)$?

2.24 Show that the time-averaged energy density and Poynting vector of the obliquely moving wave of Eq. (2.9.10) are given by

$$w = \frac{1}{2}\mathrm{Re}\Big[\frac{1}{2}\epsilon\,\boldsymbol{E}\cdot\boldsymbol{E}^* + \frac{1}{2}\mu\,\boldsymbol{H}\cdot\boldsymbol{H}^*\Big] = \frac{1}{2}\epsilon\big(|A|^2+|B|^2\big)$$
$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\mathrm{Re}\big[\boldsymbol{E}\times\boldsymbol{H}^*\big] = \hat{\mathbf{z}}'\,\frac{1}{2\eta}\big(|A|^2+|B|^2\big) = (\hat{\mathbf{z}}\cos\theta + \hat{\mathbf{x}}\sin\theta)\,\frac{1}{2\eta}\big(|A|^2+|B|^2\big)$$

where $\hat{\mathbf{z}}' = \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{x}}\sin\theta$ is the unit vector in the direction of propagation. Show that the energy transport velocity is $\boldsymbol{v} = \boldsymbol{\mathcal{P}}/w = c\,\hat{\mathbf{z}}'$.

2.25 A uniform plane wave propagating in empty space has electric field:

$$\boldsymbol{E}(x,z,t) = \hat{\mathbf{y}}\,E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}\,,\qquad k = \frac{\omega}{c_0}$$

a. Inserting $\boldsymbol{E}(x,z,t)$ into Maxwell's equations, work out an expression for the corresponding magnetic field $\boldsymbol{H}(x,z,t)$.

b. What is the direction of propagation and its unit vector $\hat{\mathbf{k}}$?

c. Working with Maxwell's equations, determine the electric field $\boldsymbol{E}(x,z,t)$ and propagation direction $\hat{\mathbf{k}}$, if we started with a magnetic field given by:

$$\boldsymbol{H}(x,z,t) = \hat{\mathbf{y}}\,H_0 e^{j\omega t}e^{-jk(\sqrt{3}z - x)/2}$$

2.26 A linearly polarized light wave with electric field $\boldsymbol{E}_0$ at angle θ with respect to the x-axis is incident on a polarizing filter, followed by an identical polarizer (the analyzer) whose primary axes are rotated by an angle ϕ relative to the axes of the first polarizer, as shown in Fig. 2.13.1.

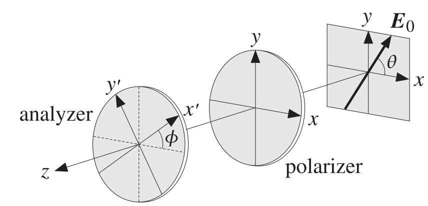


Fig. 2.13.1 Polarizer–analyzer filter combination.

Assume that the amplitude attenuations through the first polarizer are a_1, a_2 with respect to the x- and y-directions. The polarizer transmits primarily the x-polarization, so that $a_2 \ll a_1$. The analyzer is rotated by an angle ϕ so that the same gains a_1, a_2 now refer to the x'- and y'-directions.

a. Ignoring the phase retardance introduced by each polarizer, show that the polarization vectors at the input, and after the first and second polarizers, are:

$$\boldsymbol{E}_0 = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$$
$$\boldsymbol{E}_1 = \hat{\mathbf{x}}\,a_1\cos\theta + \hat{\mathbf{y}}a_2\sin\theta$$
$$\boldsymbol{E}_2 = \hat{\mathbf{x}}'\,(a_1^2\cos\phi\cos\theta + a_1a_2\sin\phi\sin\theta) + \hat{\mathbf{y}}'\,(a_2^2\cos\phi\sin\theta - a_1a_2\sin\phi\cos\theta)$$

where $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}'\}$ are related to $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ as in Problem 4.7.

b. Explain the meaning and usefulness of the matrix operations:

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}\quad\text{and}$$
$$\begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$$

c. Show that the output light intensity is proportional to the quantity:

$$\begin{aligned} I = {} & (a_1^4\cos^2\theta + a_2^4\sin^2\theta)\cos^2\phi + a_1^2a_2^2\sin^2\phi + \\ & + 2a_1a_2(a_1^2 - a_2^2)\cos\phi\sin\phi\cos\theta\sin\theta\end{aligned}$$

d. If the input light were unpolarized, that is, incoherent, show that the average of the intensity of part (c) over all angles $0 \le \theta \le 2\pi$, will be given by the generalized Malus's law:

$$\bar{I} = \frac{1}{2}(a_1^4 + a_2^4)\cos^2\phi + a_1^2a_2^2\sin^2\phi$$

The case $a_2 = 0$, represents the usual *Malus' law.*

2.27 First, prove the equivalence of the three relationships given in Eq. (2.11.11). Then, prove the following identity between the angles θ, θ':

$$(1-\beta\cos\theta)(1+\beta\cos\theta') = (1+\beta\cos\theta)(1-\beta\cos\theta') = 1-\beta^2$$

Using this identity, prove the alternative Doppler formulas (2.11.12).

2.28 In proving the relativistic Doppler formula (2.11.14), it was assumed that the plane wave was propagating in the z-direction in all three reference frames S, S_a, S_b. If in the frame S the wave is propagating along the θ-direction shown in Fig. 2.11.2, show that the Doppler formula may be written in the following equivalent forms:

$$f_b = f_a \frac{\gamma_b(1-\beta_b\cos\theta)}{\gamma_a(1-\beta_a\cos\theta)} = f_a\gamma(1-\beta\cos\theta_a) = \frac{f_a}{\gamma(1+\beta\cos\theta_b)} = f_a\sqrt{\frac{1-\beta\cos\theta_a}{1+\beta\cos\theta_b}}$$

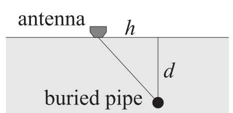
where

$$\beta_a = \frac{v_a}{c}, \quad \beta_b = \frac{v_b}{c}, \quad \beta = \frac{v}{c}, \quad \gamma_a = \frac{1}{\sqrt{1-\beta_a^2}}, \quad \gamma_b = \frac{1}{\sqrt{1-\beta_b^2}}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

and v is the relative velocity of the observer and source given by Eq. (2.11.7), and θ_a, θ_b are the propagation directions in the frames S_a, S_b. Moreover, show the following relations among these angles:

$$\cos\theta_a = \frac{\cos\theta-\beta_a}{1-\beta_a\cos\theta}, \quad \cos\theta_b = \frac{\cos\theta-\beta_b}{1-\beta_b\cos\theta}, \quad \cos\theta_b = \frac{\cos\theta_a-\beta}{1-\beta\cos\theta_a}$$

2.29 Ground-penetrating radar operating at 900 MHz is used to detect underground objects, as shown in the figure below for a buried pipe. Assume that the earth has conductivity $\sigma = 10^{-3}$ S/m, permittivity $\epsilon = 9\epsilon_0$, and permeability $\mu = \mu_0$. You may use the "weakly lossy dielectric" approximation.



a. Determine the numerical value of the wavenumber $k = \beta - j\alpha$ in meters^{-1}, and the penetration depth $\delta = 1/\alpha$ in meters.

b. Determine the value of the complex refractive index $n_c = n_r - jn_i$ of the ground at 900 MHz.

c. With reference to the above figure, explain why the electric field returning back to the radar antenna after getting reflected by the buried pipe is given by

$$\left|\frac{E_{\text{ret}}}{E_0}\right|^2 = \exp\left[-\frac{4\sqrt{h^2+d^2}}{\delta}\right]$$

where E_0 is the transmitted signal, d is the depth of the pipe, and h is the horizontal displacement of the antenna from the pipe. You may ignore the angular response of the radar antenna and assume it emits isotropically in all directions into the ground.

d. The depth d may be determined by measuring the roundtrip time $t(h)$ of the transmitted signal at successive horizontal distances h. Show that $t(h)$ is given by:

$$t(h) = \frac{2n_r}{c_0}\sqrt{d^2+h^2}$$

where n_r is the real part of the complex refractive index n_c.

e. Suppose $t(h)$ is measured over the range $-2 \le h \le 2$ meters over the pipe and its minimum recorded value is $t_{\min} = 0.2$ μsec. What is the depth d in meters?